待ち行列研究部会

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頂点が増えるグラフ上のRWのはなし

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1. 来嶋秀治, 清水伸高, 白髪丈晴, 動的グラフ上のランダムウォーク, 応用数理, 32:1 (2022), 5--15.
2. S. Kijima, N. Shimizu and T. Shiraga, How many vertices does a random walk miss in a network with a moderately increasing number of vertices?, Mathematics of Operations Research, to appear.
3. S. Kumamoto, S. Kijima and T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, LIPIcs, 302 (AofA 2024), 22:1--22:15.
4. S. Kumamoto, S. Kijima and T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, LIPIcs, 292 (SAND 2024), 17:1-17:17.

1. 背景

1.1. ランダムウォークのcover time

1.2. 動的グラフ上のRWの解析

ランダムウォーク

$$G = (V, E), P(u, v) = \Pr[X_{t+1} = v | X_t = u]$$

- **Simple RW**

$$P(u, v) = \frac{1}{\deg(u)}$$

- **Metropolis walk**

$$P(u, v) = \frac{1}{\max\{\deg(u), \deg(v)\}}$$

$$\left(= \frac{1}{\deg(u)} \min\left\{\frac{\deg(u)}{\deg(v)}, 1\right\}\right)$$

- **Lazy chain**

$$P' = \frac{P + I}{2}$$

- Biased walk, Markov chain

ランダムウォークの指標

- **Hitting time**（確率変数，しばしば期待値を指す）

  $T = \min\{t \mid X_0 = v, X_t = u\}$ 　vから出発してuに至るまでの時間

  $\tau_{\mathrm{hit}} = \mathrm{E}[T]$

  - Return time: $T = \min\{t > 0 \mid X_0 = v, X_t = v\}$
  - Commute time: $T = \min\{t \mid X_0 = u, X_t = u, \exists s < t, X_s = u\}$

- **Cover time**（確率変数，しばしば期待値を指す）

  $T = \min\{t \mid \{v \mid X_s = v\ 0 \le s \le t\} = V\}$

  $\tau_{\mathrm{cov}} = \mathrm{E}[T]$ 　vから出発して全頂点を訪問する時間

- **Mixing time**（Pに対する定数）

  $\tau(\epsilon) = \min\{t \mid \forall t' \ge t, \forall v, d_{\mathrm{TV}}(P_v^t, \pi) \le \epsilon\}$ 　定常分布に収束する時間

  - Coupling time: $T = \min\{t \mid X_t = Y_t, X_0 \ne Y_0\}$
  - Blanket time: $T(\delta) = \min\{t \mid \forall v, N_v(t) > \delta\pi_v t\}$

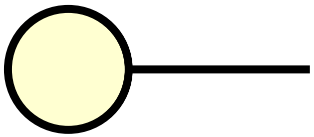
Cover time

- **Simple RW**
  - ✓ 任意のグラフに対して$\tau_{\mathrm{cov}} \leq 2m(n-1) = \mathrm{O}(n^3)$

    [Aleliunas, Karp, Lipton, Lovász, and Rackoff 1979]
  - ✓ 任意のグラフに対して

    $$(1+o(1))n\log n \leq \tau_{\mathrm{cov}} \leq \left(\frac{4}{27}+o(1)\right) n^3$$

    - ➢ 完全グラフ: $\tau_{\mathrm{cov}} \leq n(\log n + \gamma)$
    - ➢ ロリポップグラフ: $\tau_{\mathrm{cov}} \geq \left(\frac{4}{27}+o(1)\right) n^3$



[Feige 1995]× 2

- **Biased RW**
  - ✓ 任意のグラフに対してβ-RWのcover timeは$\mathrm{O}(n^2 \log n)$

    [Ikeda, Kubo, Okumoto, and Yamashita 2003]
  - ✓ 任意のグラフに対してMetropolis walkのcover time $\mathrm{O}(n^2 \log n)$

    [Nonaka, Ono, Sadakane, and Yamashita 2010]
  - ✓ 任意のグラフに対して1/min-degree RWのcover timeは$\mathrm{O}(n^2)$

    [David and Feige 2017]
- **Matthews' bound**

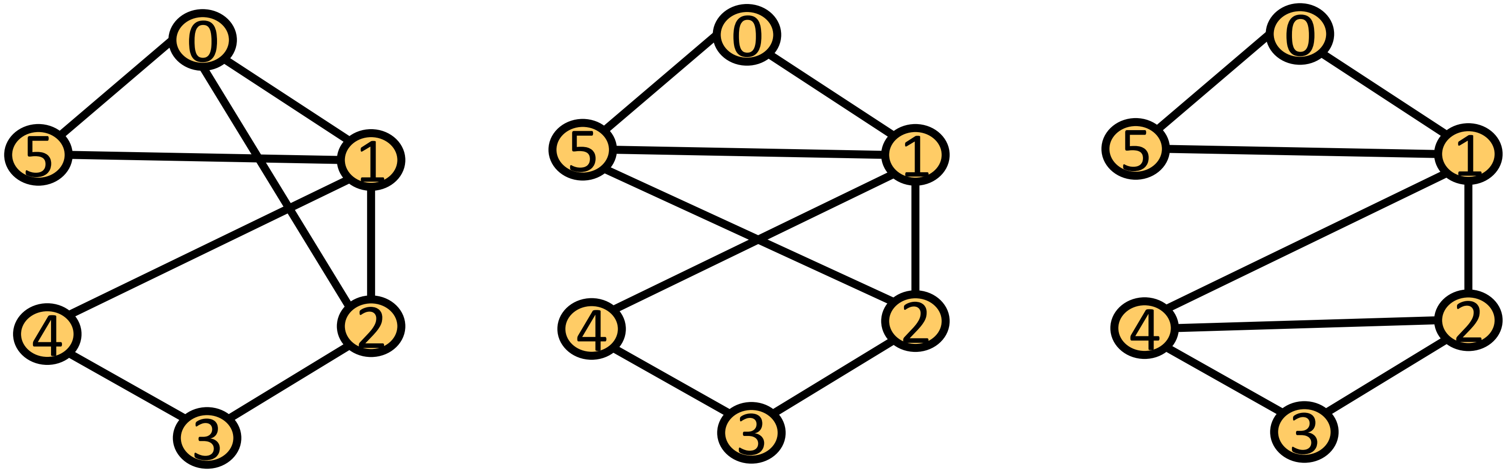
$$\tau_{\mathrm{hit}} \leq \tau_{\mathrm{cov}} \leq \tau_{\mathrm{hit}} H_n$$

[Matthews 1988]

Dynamic graph上のRWについて何がいえる？

Dynamic graph = 時間とともにグラフが変化する

$\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$　　$\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$　　$\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$



$t = 0$　　$t = 1$　　$t = 2$　　・・・

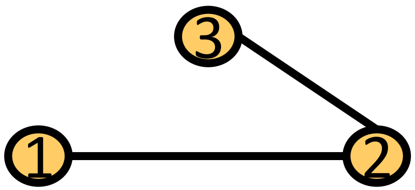
1. 背景

1.1. ランダムウォークのcover time

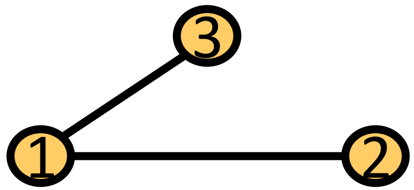
1.2. 動的グラフ上のRWの解析

“busy” simple RW on dynamic graph

- Simple RW: $P_t(u,v) = \frac{1}{\deg_t(u)} \ (v \in N(u))$
- $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$: 連結



$t \equiv 0 \pmod 2$



$t \equiv 1 \pmod 2$

観察

$X_0 = 1$とすると，頂点3を訪問できない.

⇨ RWの周期性を利用すれば，悪い例が簡単に作れる.

Lazy simple RW on dynamic graph

- Lazy simple RW

$$P(u,v)=\begin{cases}\dfrac{1}{2} & u=v\\ \dfrac{1}{2\deg(u)} & v\in N(u)\end{cases}$$

- $\mathcal{G}_t=(\mathcal{V}_t,\mathcal{E}_t)$: 連結

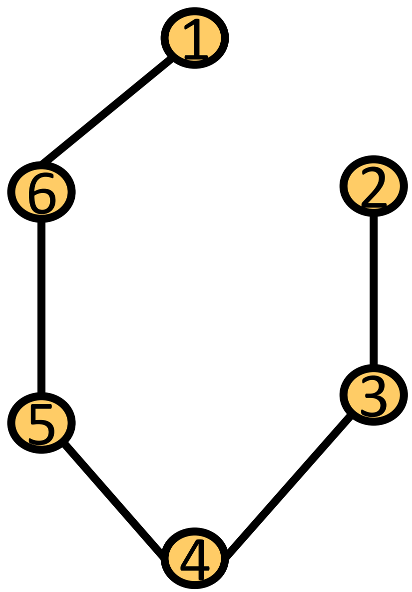
Q.
Lazy simple RWのcover time は $\mathrm{poly}(n)$?

⇨ $\mathcal{G}_t$がX_tに依存する(adaptive dynamic graph)と cover不可能.

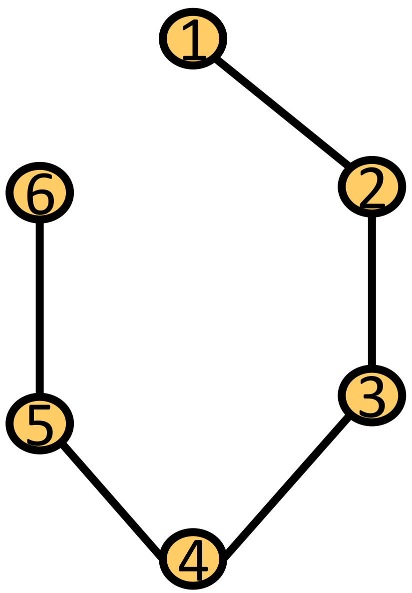
Adaptive (adversarial) dynamic graph

$$P(u,v) = \begin{cases} \frac{1}{2} & u = v \\ \frac{1}{2\deg(u)} & v \in N(u) \end{cases}$$

- $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$: 連結



$X_t \in \{1,2,3\}$のとき



$X_t \in \{4,5,6\}$のとき

観察

$X_0 = 6$とすると，頂点1を訪問できない．

Lazy simple RW on dynamic graph

- Lazy simple RW

$$P(u,v)=\begin{cases}\frac{1}{2} & u=v\\ \frac{1}{2\deg(u)} & v\in N(u)\end{cases}$$

- $\mathcal{G}_t=(\mathcal{V}_t,\mathcal{E}_t)$: 連結

Q.
Lazy simple RWのcover time は $\mathrm{poly}(n)$?

⇒ $\mathcal{G}_t$がX_tに依存する(adaptive dynamic graph)とcover不可能.

$\mathcal{G}_t$がX_tに依存しなければ？

RW on dynamic graphs

- **Cover time**
  - ✓ Lazy simple RWのcover timeは$\Omega(2^n)$
  - ✓ $P(u,v) = \frac{1}{2d_{\max}}$ ($d_{\max}$-lazy RW)は$\tau_{\mathrm{cov}} = O(n^5 (\log n)^2)$

    [Avin, Koucky, and Lotker 2008]

    - ➢ 改善 $d_{\max}$-lazy RWは$\tau_{\mathrm{cov}} = O(n^3 \log n)$

    [Denysyuk and Rodrigues 2014]
  - ✓ 各頂点の次数が不変のときlazy simple RWは$\tau_{\mathrm{cov}} = O(n^3 (\log n)^2)$

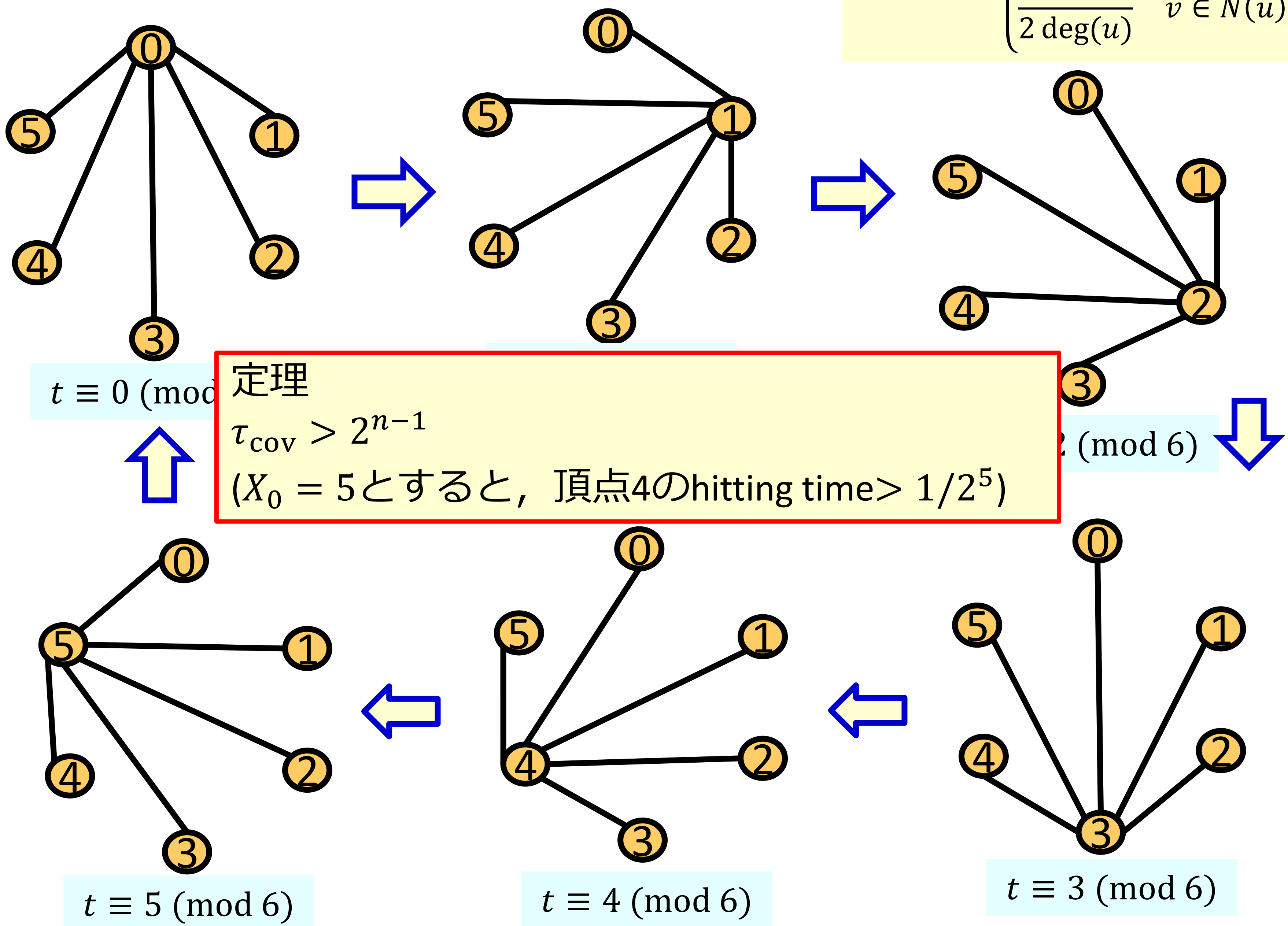
    [Sauerwald and Zanetti 2019]
- **Mixing time**
  - ✓ 定常分布が不変だと静的グラフと同様

    [Saloff-Coste and Zuniga 2011]
- **Edge Markovian Model**

  [Lamprou, Martin, and Spirakis 2018]

Sisyphus graph（シーシュポスグラフ）

$$P(u,v) = \begin{cases} \dfrac{1}{2} & u = v \\ \dfrac{1}{2\deg(u)} & v \in N(u) \end{cases}$$



定理

$\tau_{\mathrm{cov}} > 2^{n-1}$

($X_0 = 5$とすると，頂点4のhitting time$> 1/2^5$)

頂点集合が変化するグラフの場合，何がいえる？

2. 増えるクーポン収集問題

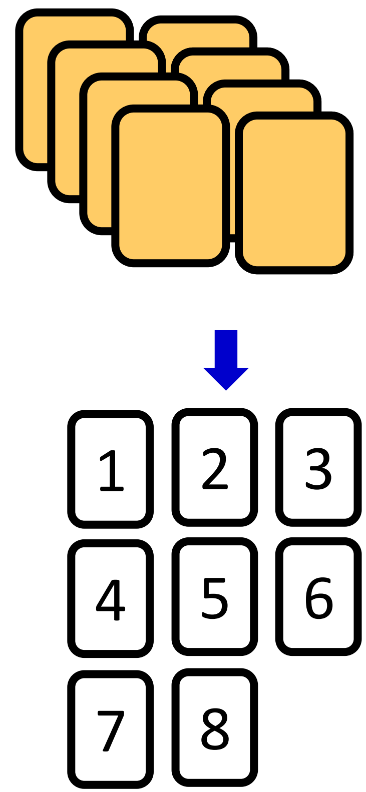
クーポン収集問題
= 完全グラフ上のRWの cover time

2. S. Kijima, N. Shimizu and T. Shiraga, How many vertices does a random walk miss in a network with a moderately increasing number of vertices?, Math OR, 2025 (to appear).

クーポン収集

- ビックリマンシール
- プロ野球チップス
- ガチャ

- P●KEM●Nカード100種類．コンプまでに何枚引けば良い？
- *n*種類のカードが裏返しにおいてある．1枚選んでめくり，種類を確認したら元に戻して，よく混ぜる．全ての種類を確認するまでに何回めくればよいか？
- 年賀状のお年玉くじ下2桁．00から99まで そろえるには年賀状を何枚もらえばよいか？
- 一万円札を千円札10枚に両替する．千円札の 記番号下2桁を00から99までそろえるには，いくら両替すればよいか？



クーポン収集

- $X_i \in \{1,\ldots,n\}$ $(i=1,2,\ldots)$は独立とし，$\Pr[X_i=k]=\frac{1}{n}(k=1,\ldots,n)$.
- $T=\min\{t \mid \{X_1,\ldots,X_t\}=\{1,\ldots,n\}\}$をコンプ回数(completion time)と呼ぶ.

Q. $\mathrm{E}[T]$を求めよ.

$T_k := \min\{t \mid |\{X_1,\ldots,X_t\}|=k\}$:$k$種コンプ回数

$S_k := T_k - T_{k-1}$:$k-1$種コンプしてから新種が出るまでの回数

Claim. $\mathrm{E}[S_k]=\frac{n}{n-k+1}$.

- $k-1$種コンプした状態で，新種の出る確率$p_k=\frac{n-(k-1)}{n}$.
- 従って$\mathrm{E}[S_k]=\frac{1}{p_k}=\frac{n}{n-k+1}$ (幾何分布の期待値).

$T=T_n=\sum_{k=1}^n S_k$に注意して，

$$\mathrm{E}[T]=\mathrm{E}[T_n]=\sum_{k=1}^n \mathrm{E}[S_k]=\sum_{k=1}^n \frac{n}{n-k+1}=n\sum_{k'=1}^n \frac{1}{k'} \simeq n(\ln n+0.577)$$

$\simeq 519$ ($n=100$の時)

クーポン収集

- ✓ $X_i \in \{1,\dots,n\}$ $(i = 1,2,\dots)$は独立とし，$\Pr[X_i = k] = \frac{1}{n}(k = 1,\dots,n)$.
- ✓ $T = \min\{t \mid \{X_1,\dots,X_t\} = \{1,\dots,n\}\}$をコンプ回数(completion time)と呼ぶ.

Q. $\mathrm{E}[T]$を求めよ.

Q. $\Pr[T \geq 2n\ln n]$はどれくらいか？

Chevyshevの不等式を使うと

$$\Pr\left[X \geq 2\mathrm{E}[X]\right] \leq \frac{\mathrm{Var}[X]}{\mathrm{E}[X]^2} \leq \frac{\frac{n^2\pi^2}{6}}{(n\ln n)^2} = \frac{\pi^2}{6(\ln n)^2}$$

$\simeq 0.078$ ($n = 100$の時)

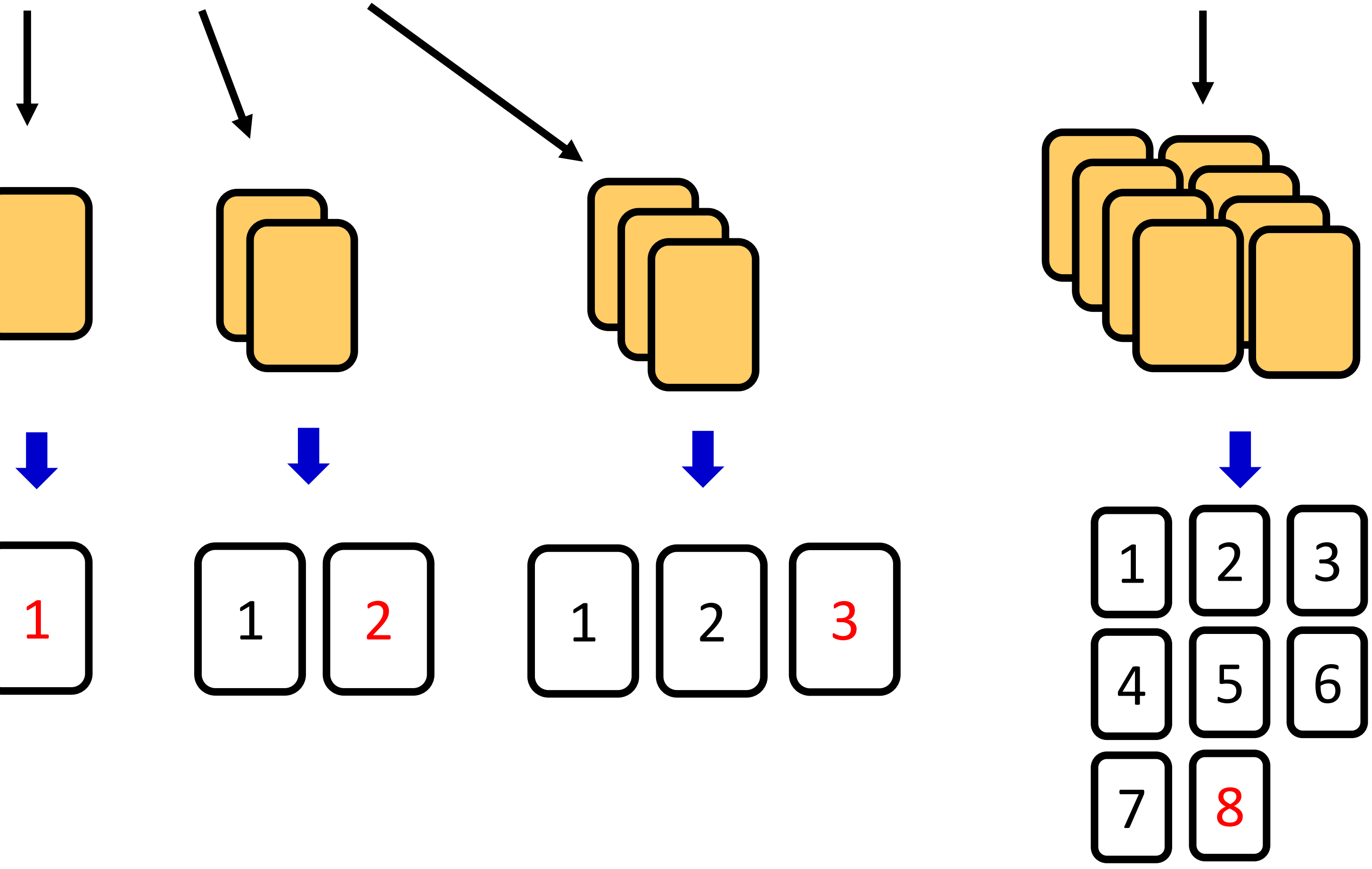
増えるクーポン収集

毎日1回ガチャを引く.

- 毎日1種ずつ増えるとき，コンプ回数は？

増えるクーポン収集

Day (i)	1日目	2日目	3日目	4日目	5日目	6日目	7日目	8日目	9日目
種類	1	2	3	4	5	6	7	8	9
$\Pr[X_i = k]$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$



増えるクーポン収集

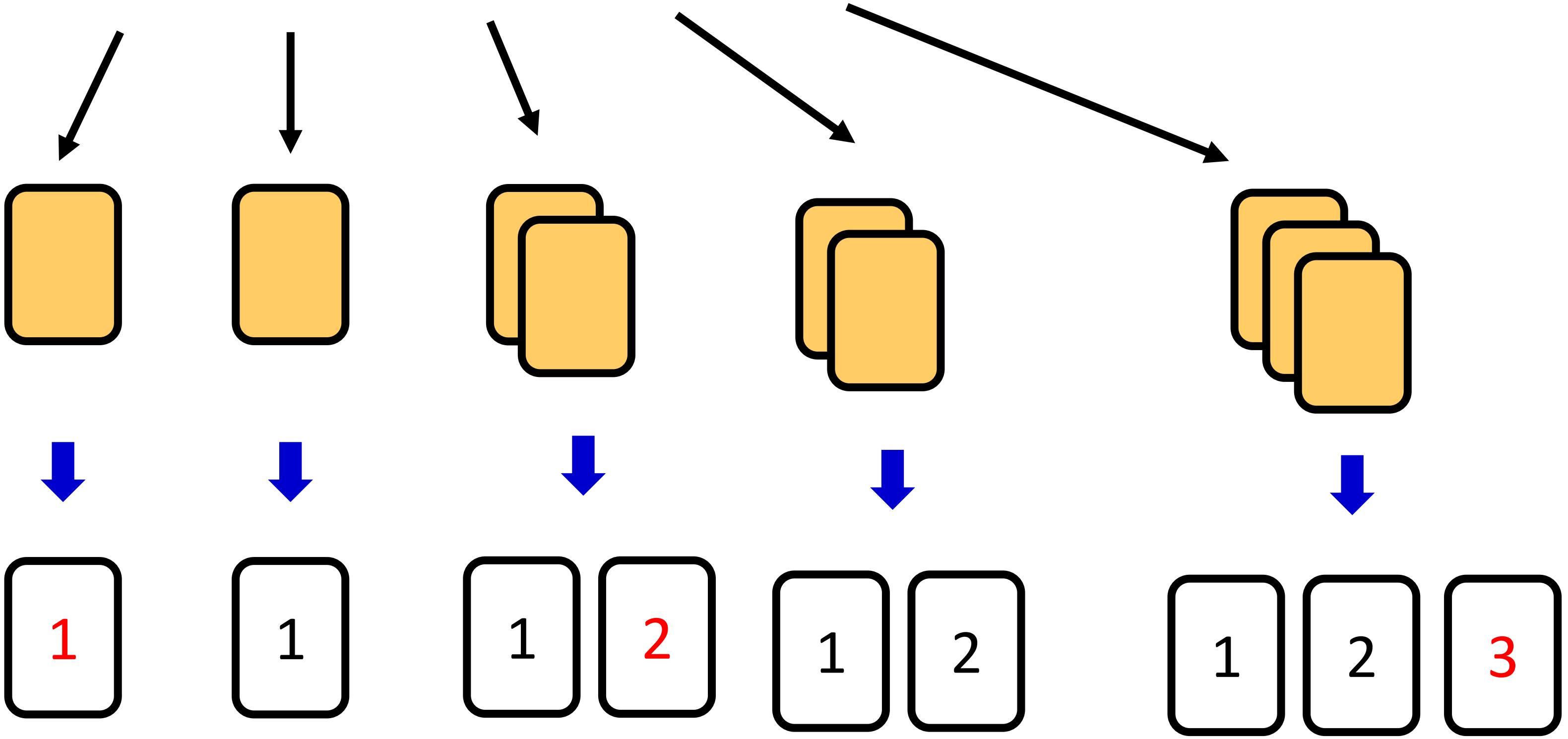
毎日1回ガチャを引く．

- 毎日1種ずつ増えるとき，コンプ回数は？
  - コンプ不可能

- 2日に1回新種リリースされたら，コンプ回数は？

増えるクーポン収集

Day	1日目	2日目	3日目	4日目	5日目	6日目	7日目	8日目	9日目
種類	1	1	2	2	3	3	4	4	5
$\Pr[X_i = k]$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{5}$



増えるクーポン収集

毎日1回ガチャを引く．

- 毎日1種ずつ増えるとき，コンプ回数は？
  - コンプ不可能

- 2日に1回新種リリースされたら，コンプ回数は？
  - コンプ不可能

- 10日に1回リリースされたら？
  - コンプ不可能(種類数1000の時を考えてみよ)

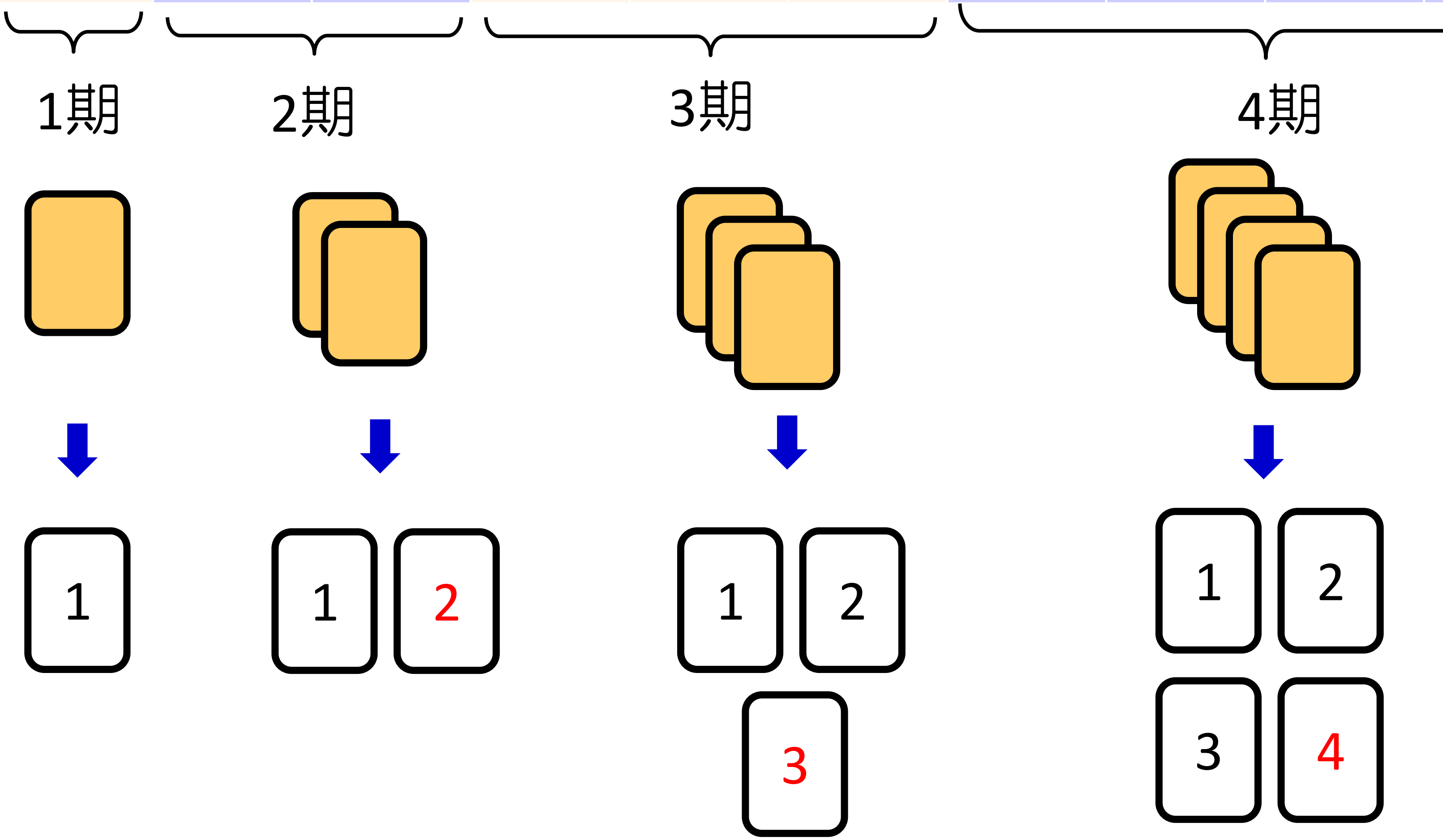
ゆっくりと増えるクーポン収集

毎日1回ガチャを引く.

- n種類目がリリースされてn日後に新種がリリース.
  - Q. コンプ回数は？

ゆっくりと増えるクーポン収集

Day	1日目	2日目	3日目	4日目	5日目	6日目	7日目	8日目	9日目
種類	1	2	2	3	3	3	4	4	4
$\Pr[X_i = k]$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$



ゆっくりと増えるクーポン収集

毎日1回ガチャを引く.

- n種類目がリリースされてn日後に新種がリリース.
  - Q. コンプ回数は？
    - ✓ A. コンプ不可能(新種は出続けるから)

- **Q. n期末にはどのくらいコンプされる？**
  1. $o(n)$
  2. cn
  3. $n - c\sqrt{n}$
  4. $n - c\log n$
  5. $n - c$

cは適当な定数

ゆっくりと増えるクーポン収集

毎日1回ガチャを引く(一様ランダム)
$\mathfrak{d}(n)$: n期の日数
U_n: n期末の未収集アイテムの種類数

命題

$\mathfrak{d}(n) = n$のとき，$\mathrm{E}[U_n] < 1$.

証明

- ✓ $\mathcal{E}_{i,n} := \begin{cases} 1 & (n\text{期末にアイテム } i \text{ が未収集}) \\ 0 & (n\text{期末にアイテム } i \text{ が収集済み}) \end{cases} \qquad (i = 1, 2, \dots, n)$
- ✓ $U_n = \sum_{i=1}^{n} \mathcal{E}_{i,n}$
- ✓ アイテムnがn期末に未収集の確率

$$\Pr[\mathcal{E}_{n,n} = 1] = \left(1 - \frac{1}{n}\right)^n < \mathrm{e}^{-1}$$

- ✓ アイテム i ($i \le n$)が n期末に未収集の確率

$$\Pr\left[\mathcal{E}_{i,n} = 1\right] = \left(1 - \frac{1}{i}\right)^i \left(1 - \frac{1}{i+1}\right)^{i+1} \dots \left(1 - \frac{1}{n}\right)^n < \left(\frac{1}{\mathrm{e}}\right)^{n+1-i}$$

- ✓ $\mathrm{E}[U_n] = \sum_{i=1}^{n} \Pr\left[\mathcal{E}_{i,n}\right] < \sum_{i=1}^{n} \left(\frac{1}{\mathrm{e}}\right)^{n+1-i} = \frac{1}{\mathrm{e}} + \frac{1}{\mathrm{e}^2} + \dots + \frac{1}{\mathrm{e}^n} < \frac{\frac{1}{\mathrm{e}}}{1 - \frac{1}{\mathrm{e}}} = \frac{1}{\mathrm{e}-1} < 0.582$

ゆっくりと増える クーポン下界

毎日1回ガチャを引く(一様ランダム)

$\mathfrak{d}(n)$: n期の日数

U_n: n期末の未収集アイテムの種類数

命題

$\mathfrak{d}(n) = cn$のとき, $\mathrm{E}[U_n] \geq \frac{n}{cn+1}\left(1-\frac{1}{n}\right)^{cn} \simeq \frac{1}{ce^c}$.

証明

- ✓ $S(n) \coloneqq \sum_{k=1}^{n} \prod_{i=k}^{n} \left(1-\frac{1}{i}\right)^{ci}$ $\quad (= \mathrm{E}[U_n])$
- ✓ Proposition $S(n) \geq \frac{n}{cn+1}\left(1-\frac{1}{n}\right)^{cn}$.
  - Claim 1: $S(n+1) = \left(1-\frac{1}{n+1}\right)^{c(n+1)} (S(n)+1)$.
  - Claim 2: $S(n) \geq \frac{n}{cn+1}\left(1-\frac{1}{n}\right)^{cn} \Rightarrow S(n)+1 \geq \frac{n+1}{c(n+1)+1}$.
  - 帰納法として,

$$
\begin{aligned}
S(n+1) &= \left(1-\frac{1}{n+1}\right)^{c(n+1)} (S(n)+1) \\
&\geq \left(1-\frac{1}{n+1}\right)^{c(n+1)} \frac{n+1}{c(n+1)+1}
\end{aligned}
$$

- Proof of Claim 1

$$
\begin{aligned}
S(n+1) &= \sum_{k=1}^{n+1} \prod_{i=k}^{n+1} \left(1 - \frac{1}{i}\right)^{ci} \\
&= \sum_{k=1}^{n} \prod_{i=k}^{n+1} \left(1 - \frac{1}{i}\right)^{ci} + \left(1 - \frac{1}{n+1}\right)^{c(n+1)} \\
&= \sum_{k=1}^{n} \left(1 - \frac{1}{n+1}\right)^{c(n+1)} \prod_{i=k}^{n} \left(1 - \frac{1}{i}\right)^{ci} + \left(1 - \frac{1}{n+1}\right)^{c(n+1)} \\
&= \left(1 - \frac{1}{n+1}\right)^{c(n+1)} \left(\sum_{k=1}^{n} \prod_{i=k}^{n} \left(1 - \frac{1}{i}\right)^{ci} + 1\right) \\
&= \left(1 - \frac{1}{n+1}\right)^{c(n+1)} (S(n) + 1)
\end{aligned}
$$

- Proof of Claim 2

$$
\begin{aligned}
S(n) + 1 &\geq \frac{n}{cn+1} \left(1 - \frac{1}{n}\right)^{cn} + 1 \\
&\geq \frac{n}{cn+1} \left(1 - \frac{cn}{n}\right) + 1 \\
&= \frac{n}{cn+1} \frac{n - cn}{n} + 1 = \frac{n - cn}{cn+1} + 1 = \frac{n+1}{cn+1} \geq \frac{n+1}{c(n+1)+1}
\end{aligned}
$$

ゆっくりと増えるクーポン収集

補題(S_nの性質)

$\mathfrak{d}\colon \mathbb{N} \to \mathbb{N}$, $S(n) \coloneqq \sum_{k=1}^{n} \prod_{i=k}^{n} \left(1 - \frac{1}{i}\right)^{\mathfrak{d}(i)}$

(i) $\mathfrak{d}(i) \geq ci$ のとき，$S(n) = O(1)$.

(ii) $\mathfrak{d}$が**単調非減少**($\mathfrak{d}(i) \leq \mathfrak{d}(i+1)$)のとき，

$$S(n) \geq \frac{n}{\mathfrak{d}(n)+1}\left(1 - \frac{1}{n}\right)^{\mathfrak{d}(n)}.$$

(iii) $\mathfrak{d}$が**劣線形**($\frac{\mathfrak{d}(i)}{i} \geq \frac{\mathfrak{d}(i+1)}{i+1}$)のとき，$S(n) \leq \frac{n}{\mathfrak{d}(n)}$.

(iv) $\mathfrak{d}(i) = c$ (定数)のとき，$S(n) \leq \frac{n}{c+1}$.

増えるクーポン収集

毎日1回ガチャを引く(一様ランダム)
$\mathfrak{d}(n)$: n期の日数
U_n: n期末の未収集アイテムの種類数

定理

$\mathfrak{d}: \mathbb{N} \to \mathbb{N}$.

(i) $\mathfrak{d}(i) \geq ci$ のとき, $\mathrm{E}[U_n] = \mathrm{O}(1)$.

- とくに $\frac{\mathfrak{d}(i)}{i} \xrightarrow{i\to\infty} \infty$ のとき, $\mathrm{E}[U_n] = 0$.

(ii) $\mathfrak{d}$が単調非減少で非有界かつ劣線形($\frac{\mathfrak{d}(i)}{i} \geq \frac{\mathfrak{d}(i+1)}{i+1}$)のとき,

$$E[U_n] = \left(1 - \mathrm{o}(1)\right)\frac{n}{\mathfrak{d}(n)+1}.$$

(iii) $\mathfrak{d}(i) = c$(定数)のとき,

$$E[U_n] = \left(1 - \mathrm{O}\left(\frac{1}{n}\right)\right)\frac{n}{c+1}.$$

$\mathfrak{d}(i) = \mathrm{o}(i)$の時, $E[U_n] \xrightarrow{n\to\infty} \infty$.
(e.g., $\mathfrak{d}(i) = \lceil\sqrt{i}\rceil$,
$\mathfrak{d}(i) = \lceil\log i\rceil$ etc.)

3. 頂点が増えるグラフ上のRWの “cover time”

クーポン収集から有限グラフ上のRWへ

2. S. Kijima, N. Shimizu and T. Shiraga, How many vertices does a random walk miss in a network with a moderately increasing number of vertices?, Math OR, 2025 (to appear).

Random Walk on a Growing Graph (RWoGG)モデル

[K, Shimizu, Shiraga '21]

- **Growing graph**: (静的な)グラフの列

$$\boldsymbol{\mathcal{G}} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots$$

$\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$は静的なグラフ.

便宜上，$\mathcal{V}_t \subseteq \mathcal{V}_{t+1}$を仮定.

この結果では$\mathcal{E}_t \subseteq \mathcal{E}_{t+1}$ も仮定.

- **RWoGG** $(\mathfrak{d}, G, P)$: $X_t (t = 0,1,2, \dots) \in V^{(t)}$
  - $\mathfrak{d}(1), \mathfrak{d}(2), \mathfrak{d}(3), \dots \in \mathbb{Z}$ は duration timeを表す.
  - $T_n = \sum_{i=1}^{n} \mathfrak{d}(n)$とし，Growing graph は

$$\mathcal{G}_t = G^{(n)} \quad \text{for} \quad t \in [T_{n-1}, T_{n-1} + \mathfrak{d}(n))$$

とする，つまり

$$\mathcal{G}_t = \begin{cases} G^{(1)} & \text{最初の}\ \mathfrak{d}(1)\text{ステップ} \\ G^{(2)} & \text{次の}\ \mathfrak{d}(2)\text{ステップ} \\ G^{(3)} & \text{次の}\ \mathfrak{d}(3)\text{ステップ} \\ \vdots & \vdots \end{cases}$$

  - $P^{(n)}$ は$G^{(n)}$上の推移確率行列.

$\mathfrak{d}$ は成長スピード（の逆数）を表す.

$\mathfrak{d}(n)$: n期の日数

U_n: n期末の未訪問頂点数

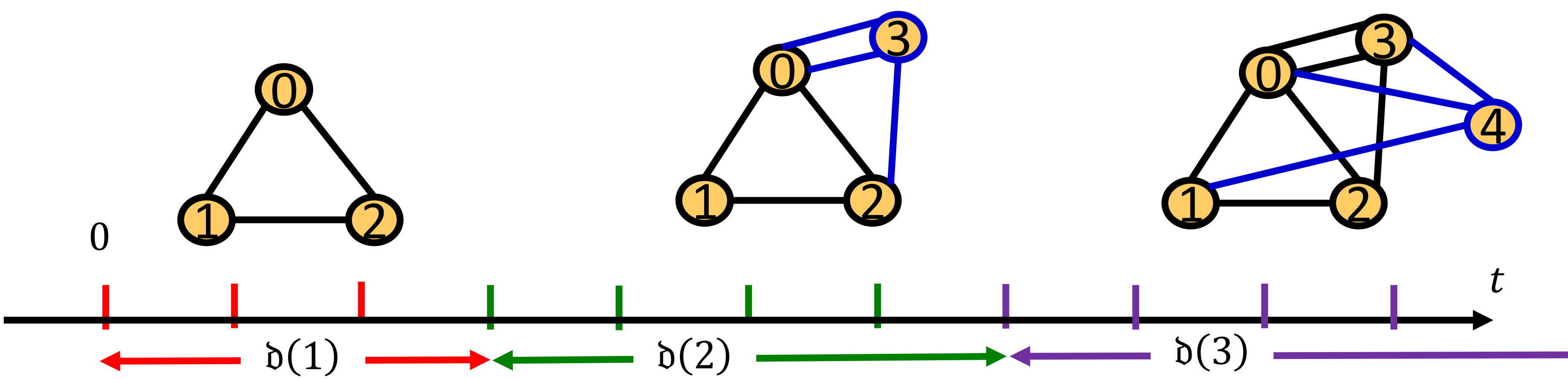
例 Preferential attachment $PA(d)$のRWoGG

$G^{(i)} = \left(V^{(i)}, E^{(i)}\right)$は再帰的に構成される:

（重複を許す．）

1. 頂点 i を追加する.
2. $\deg_{i-1}(u)$に比例する確率でd個の頂点$X_1, \dots, X_d \in V_{i-1}$を選ぶ
3. d本の枝$\{i+1, X_1\}, \dots, \{i+1, X_d\}$を追加する.

$G^{(1)} = \left(V^{(1)}, E^{(1)}\right)$ $G^{(2)} = \left(V^{(2)}, E^{(2)}\right)$ $G^{(3)} = \left(V^{(3)}, E^{(3)}\right)$



定理 (Preferential attachment)

証明略

$P^{(i)}$はlazy simpleとする． $\forall \gamma > 0, \exists C > 0,$

$\mathfrak{d}(i) \geq C i^{1-\gamma}$のとき，確率$1 - O(n^{-1})$で$\mathrm{E}[U_n] \leq 4n^{\gamma}$.

頂点の増えるグラフ上のRW

$\mathfrak{d}(n)$: n期の日数

U_n: n期末の未訪問頂点数

定理 (一般上界)

$\mathfrak{d}(i) \geq c t_{\mathrm{hit}}(i)$ $(c > 1)$のとき，$\mathrm{E}[U] = O(1)$.

さらに$\frac{\mathfrak{d}(i)}{t_{hit}(i)} \xrightarrow{i\to\infty} \infty$なら，$\mathrm{E}[U_n] \xrightarrow{n\to\infty} 0$.

証明の概略

直感的に

$$\mathrm{E}[U_n] = \sum_{k}^{n} \Pr[v_k \text{ is unvisited}] \simeq \sum_{k=1}^{n} \prod_{i=k}^{n} \Pr[v_k \text{ is unvisited @ } i^{\text{th}} \text{ period}]$$

厳密には

$$\mathrm{E}[U_n] \leq \sum_{k=1}^{n} \prod_{i=k}^{n} \Pr[T(i) > \mathfrak{d}(i)]$$

$$T(i) = \max_{u} \min\{t | X_t = v, X_0 = u\} \text{ on } G^{(i)}$$

マルコフの不等式より$\Pr[T_{\mathrm{hit}}(i) > \mathfrak{d}(i)] \leq \frac{t_{\mathrm{hit}}}{\mathfrak{d}(i)}$を代入して

$$\mathrm{E}[U_n] \leq \sum_{k=1}^{n} \prod_{i=k}^{n} \frac{t_{\mathrm{hit}}}{\mathfrak{d}(i)} \leq \sum_{k=1}^{n} \prod_{i=k}^{n} \frac{1}{c} = \sum_{k=1}^{n} \frac{1}{c^{n-k+1}} \leq \frac{1}{c-1}$$

頂点の増えるグラフ上のRW

$\mathfrak{d}(n)$: n期の日数

U_n: n期末の未訪問頂点数

定理 (lazy reversible)

$P^{(i)}$はlazy reversibleとする．

$\mathfrak{d}(i) \geq \frac{t_{\mathrm{hit}}(i)}{N} + 2t_{\mathrm{mix}}(i)$ のとき，$\mathrm{E}[U_n] \leq 8N + 32$.

証明の雰囲気

$T_\pi(i) = \max_u \min\{t | X_t = v, X_0 \sim \pi\}$ on $G^{(i)}$

- Lazy reversibleに対して$\Pr[T_\pi(i) > t] \leq \left(1 - \frac{1}{t_{\mathrm{hit}}}\right)^t \leq \exp\left(-\frac{t}{t_{\mathrm{hit}}}\right)$ by [Oliveira Peres 2019]
- $\mathrm{E}[U_n] \leq \sum_{k=1}^n \prod_{i=k}^n \Pr[T_{\mathrm{hit}}(i) > \mathfrak{d}(i)]$を上から押さえたい．

$$\Pr[T_{\mathrm{hit}}(i) > \mathfrak{d}(i)] = \sum_v \Pr\left[X_{2t_{\mathrm{mix}}} = v\right] \Pr\left[T_{\mathrm{hit}}(i) > \mathfrak{d}(i) \middle| X_{2t_{\mathrm{mix}}} = v\right]$$

$$\leq \Pr\left[T_\pi(i) > \frac{t_{\mathrm{hit}}(i)}{N}\right] + \frac{3}{4} \leq \frac{1}{4}\exp\left(-\frac{1}{N}\right) + \frac{3}{4}$$

- $\mathrm{E}[U_n] \leq \sum_{k=1}^n \left(\frac{1}{4}\exp\left(-\frac{1}{N}\right) + \frac{3}{4}\right)^{n-k+1} \leq \sum_{k=1}^n \left(\exp\left(-\frac{k}{32}\right) + \exp\left(-\frac{k}{8N}\right)\right) \leq 32 + 8N$

4. 頂点が増えるグラフ上のRWの再帰性

3. S. Kumamoto, S. Kijima and T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, LIPIcs, 302 (AofA 2024), 22:1--22:15.
4. S. Kumamoto, S. Kijima and T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, LIPIcs, 292 (SAND 2024), 17:1-17:17.

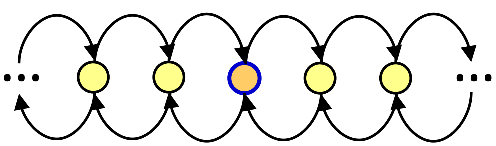
Recurrence/Transience of Random walks on *infinite* graphs

A random walk on an infinite graph is **recurrent** at vertex v if it visits v infinitely many times, i.e.,

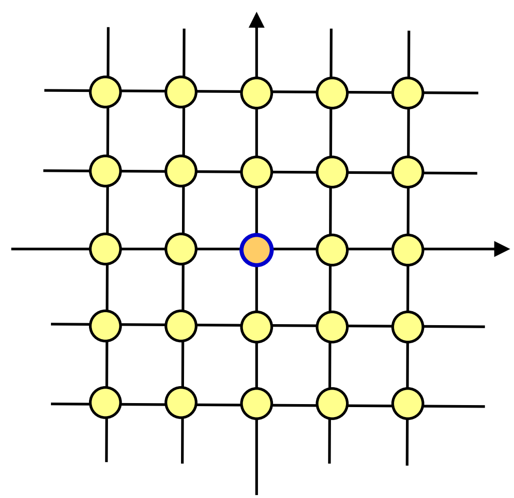
$$\sum_{t=0}^{\infty} \Pr[X_t = v] = \infty$$

holds, otherwise it is said to be **transient**.

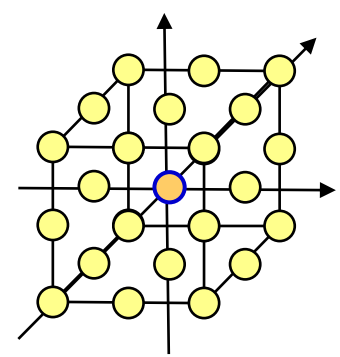
For instance,



RW on $\mathbb{Z}$ is recurrent at o,



RW on $\mathbb{Z}^2$ is recurrent at o,



RW on $\mathbb{Z}^3$ is transient at o,

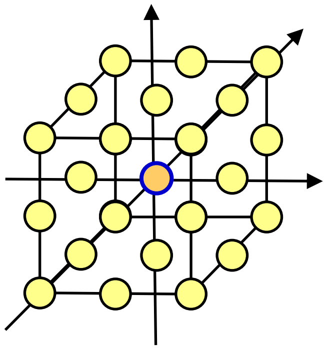
Example 1. Random walk in a growing region of $\mathbb{Z}^3$

- ✓ Random walk on $\mathbb{Z}^3$ is transient at o.
- ✓ Random walk on $\{-n, \dots, n\}^3$ is recurrent at o.

Q. Is a random walk on $\{-n, \dots, n\}^3$ recurrent or transient if n **increases** as time go on?

A. It depends on the increasing speed.

⇨ Find the phase transition point regarding the growing speed.



RW on $\mathbb{Z}^3$ is transient at o,

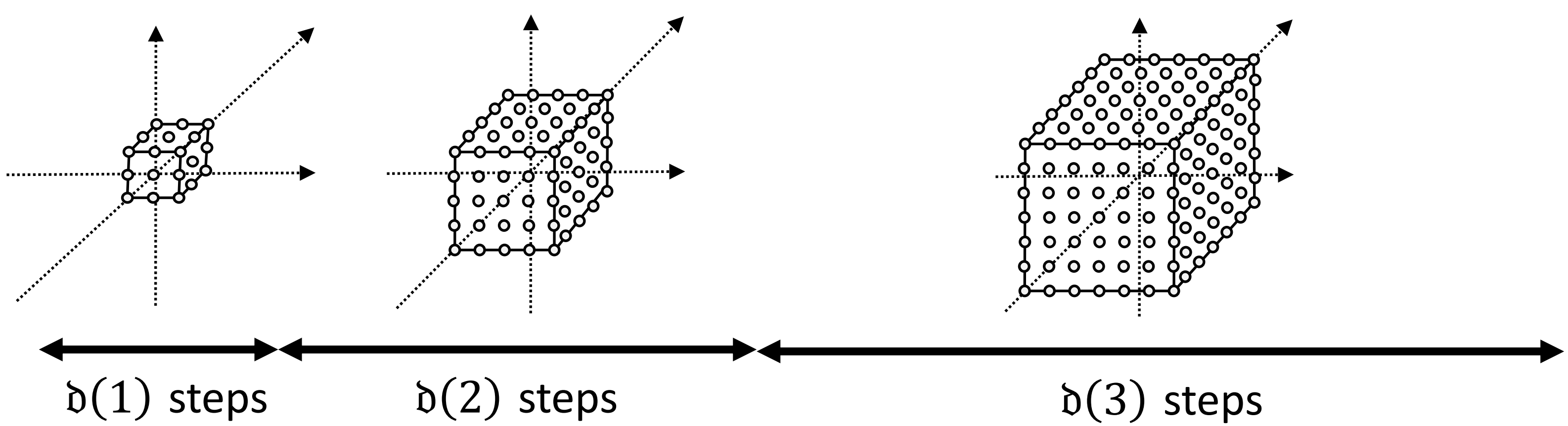
Example 1. Random walk in a growing region of $\mathbb{Z}^d$

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = n^2$,
- $G(n)$ is a grid graph $\{-n, \ldots, n\}^3$,
- $P(n)$ denotes the simple random walk w/ reflection bound,

  i.e., move to a neighbor w.p. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ unless boundary,

for $n = 1,2, \ldots$



Thm. [Dembo et al. 2014, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{n^d} = \infty$ then recurrent, otherwise transient.

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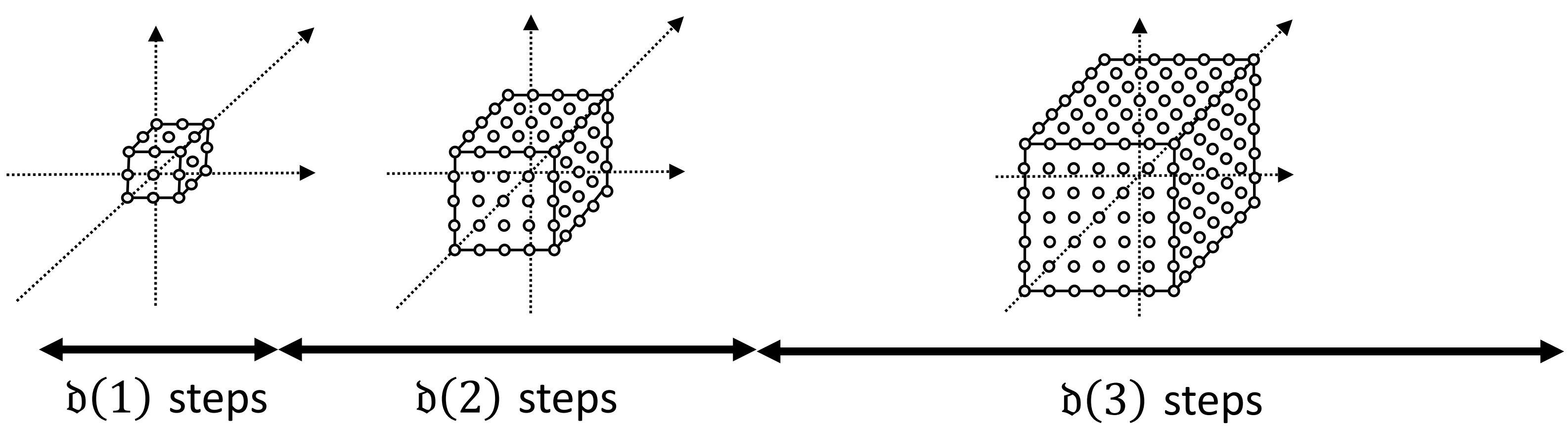
for $n = 1,2, \dots$

Recurrent

since $\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.



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Example 1. Random walk in a growing region of $\mathbb{Z}^d$

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = n^{1.999}$,
- $G(n)$ is a grid graph $\{-n, \ldots, n\}^3$,
- $P(n)$ denotes the simple random walk w/ reflection bound,

  i.e., move to a neighbor w.p. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ unless boundary,

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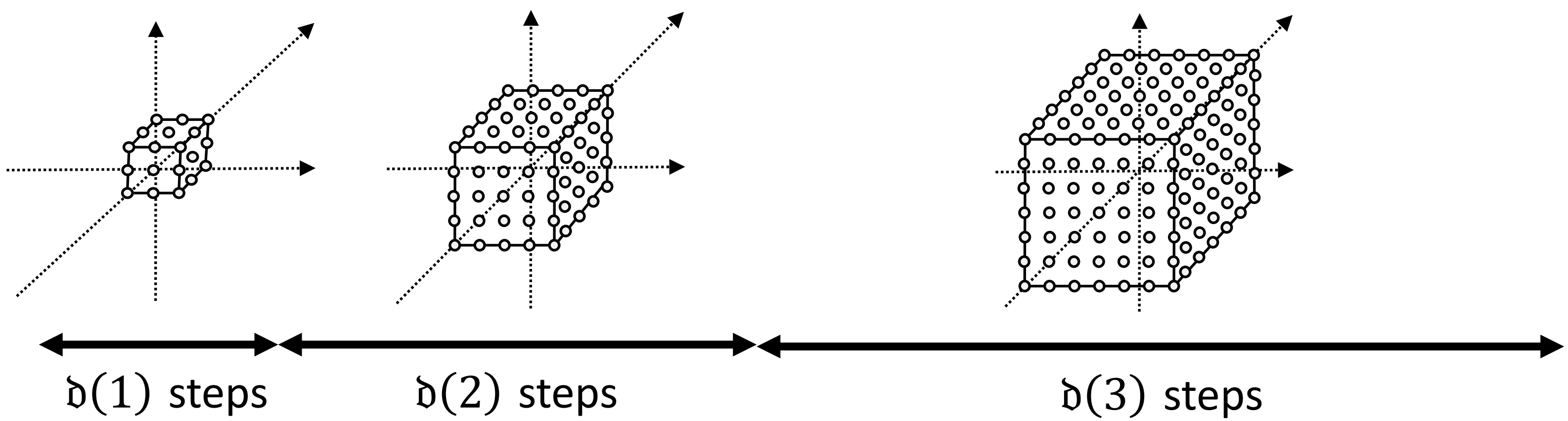

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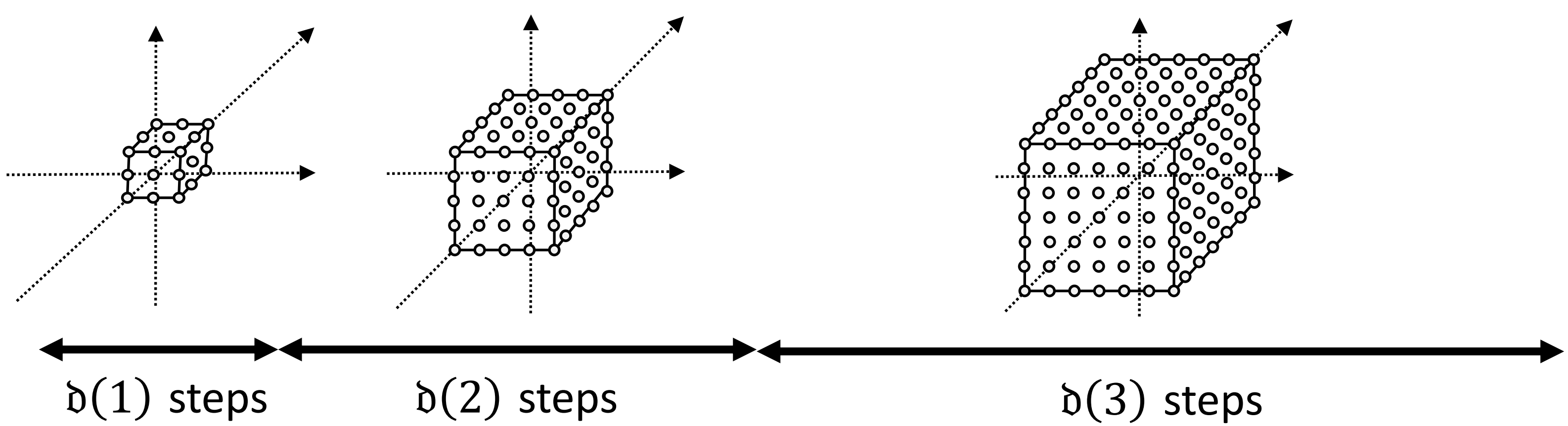
for $n = 1,2, \dots$

Transient

since $\sum_{n=1}^{\infty} \frac{n^{1.999}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{0.999}} < 1000.$



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If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{n^d} = \infty$ then recurrent, otherwise transient.

Related work (2/2): recurrence/transience of RW

- Much work about the recurrence/transience on growing graphs exist in the context of self-interacting random walks including reinforced random walks, excited random walks, etc. since 1990s, or before.
- Dembo, Huang and Sidoravicius (2014× 2): recurrent $\Leftrightarrow \sum_{t=0}^{\infty} \pi_t(0) = \infty$ for growing subregion of $\mathbb{Z}^d$ (fixed d), by conductance argument.
  - ➢ See also Huang and Kumagai (2016), Dembo, Huang, Morris and Peres (2017), Dembo, Huang and Zheng (2019), etc. about heat kernel, evolving set arguments.
- Amir, Benjamini, Gurel-Gurevich and Kozma (2015): random walk on growing tree. (random walk in changing environment).
- Huang (2017): growing graph w/ *uniformly bounded degrees*.
- Kumamoto, K. and Shirai (2024): k-ary tree, $\{0,1\}^n$ w/ an increasing n under **RWoGG** model by coupling.
- This work (2024): $\{0,1, \dots, N\}^n$ (fixed N, increasing n) by pausing coupling.

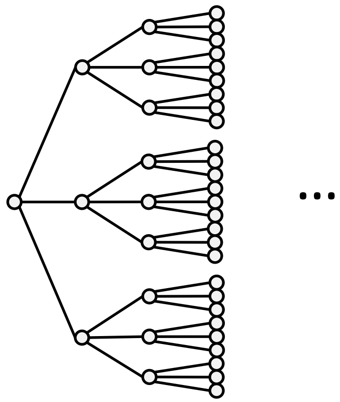
Example 2. RW on an infinite k-ary tree

- ✓ Random walk on an infinite k-ary tree is transient at r.
- ✓ Random walk on a finite k-ary tree is recurrent at r.

Q. Is a random walk on a k-ary tree recurrent or transient if its height n **increases** as time go on?

A. It depends on the increasing speed.

⇨ Find the phase transition point regarding the growing speed.



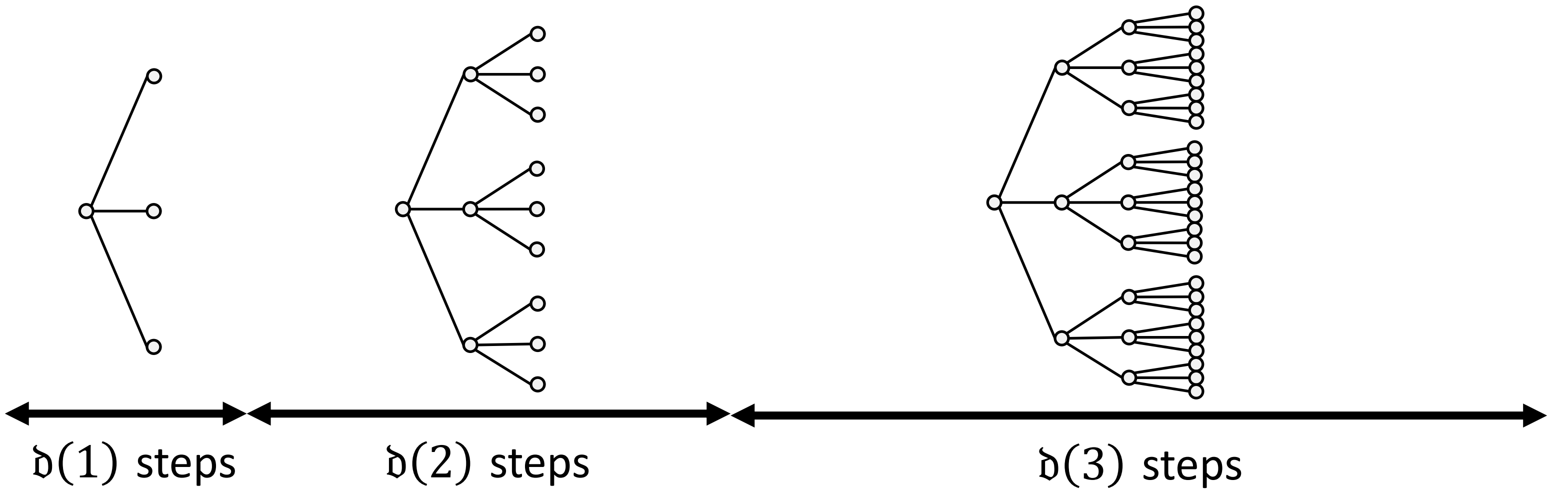
Example 2. Random walk on a growing k-ary tree

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where
- $\mathfrak{d}(n) = 3^n$,
- $G(n)$ is a 3-ary tree of height n,
- $P(n)$ denotes the simple random walk w/ reflection bound,
  i.e., move to a neighbor w.p. 1/4 unless the root or a leaf,

for $n = 1,2, \ldots$

Recurrent

since $\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1 = \infty$.



Thm. [Huang 2019, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{k^n} = \infty$ then recurrent, otherwise transient.

Example 2. Random walk on a growing k-ary tree

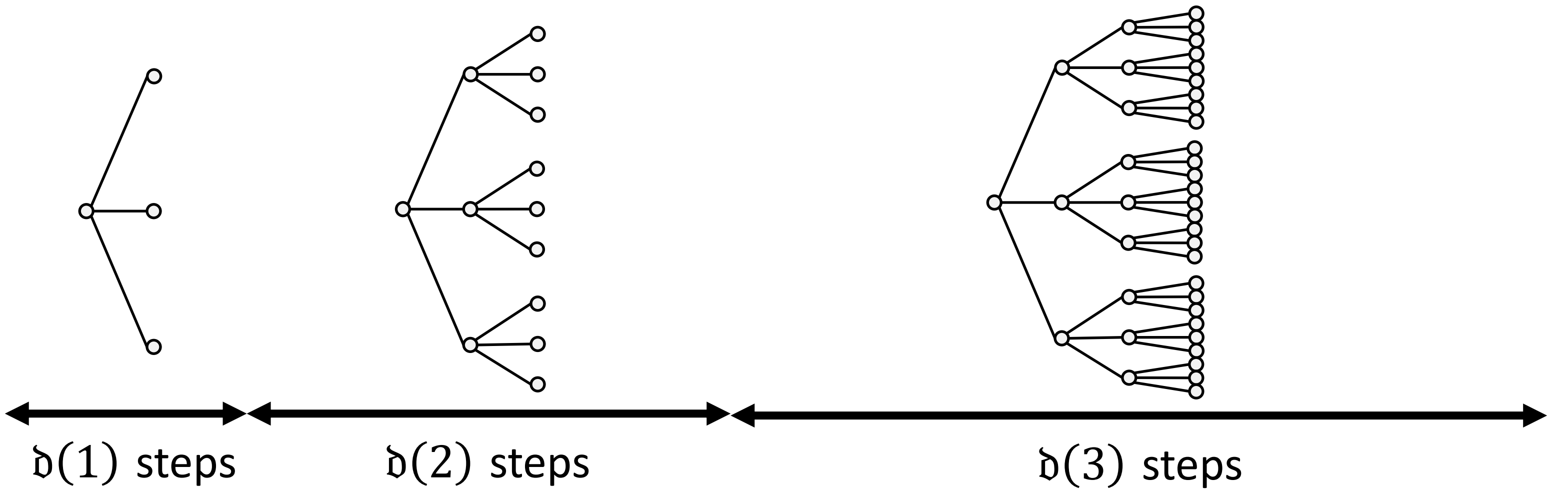
Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = 2.999999^n$,
- $G(n)$ is a 3-ary tree of height n,
- $P(n)$ denotes the simple random walk w/ reflection bound, i.e., move to a neighbor w.p. 1/4 unless the root or a leaf,

for $n = 1,2,\ldots$

Transient

since $\sum_{n=1}^{\infty} \frac{2.999999^n}{3^n} < 1{,}000{,}000$.



Thm. [Huang 2019, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{k^n} = \infty$ then recurrent, otherwise transient.

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

[SAND '24]

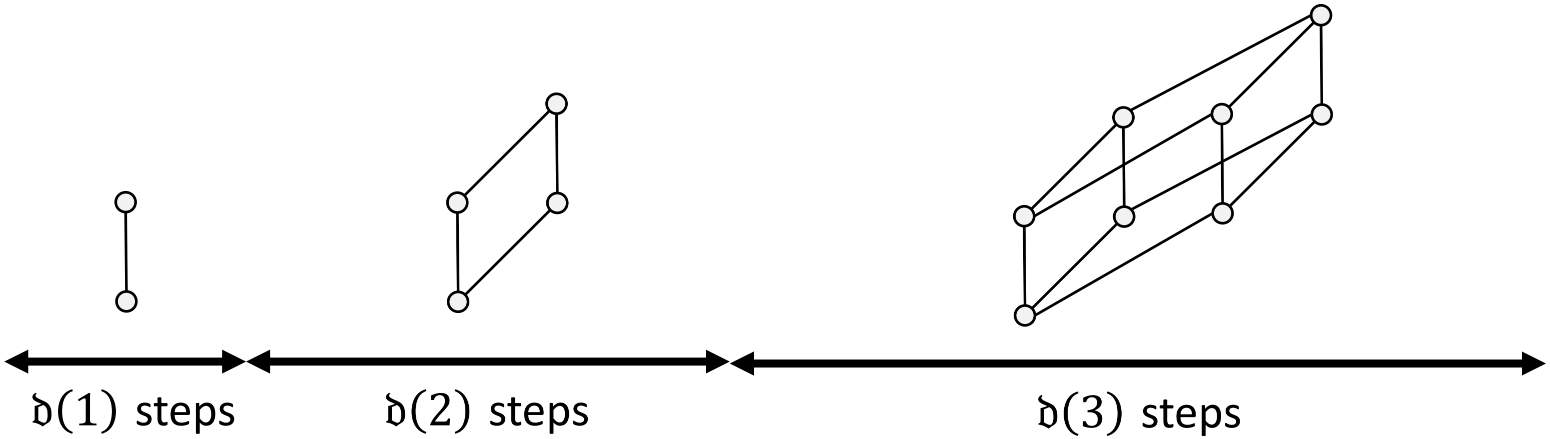
Recurrent

since $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$.

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = 2^n$,
- $G(n)$ is a $\{0,1\}^n$ skeletone,
- $P(n)$ denotes the simple random walk,
  i.e., move to a neighbor w.p. $1/n$,

for $n = 1,2,\ldots$



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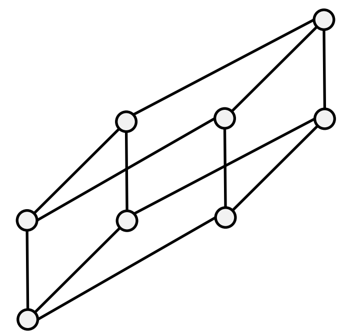
Lem. [Kumamoto et al. 2024]

Random walk on $\{0,1\}^n$ is ***LHaGG***.



$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

Thm. [Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

LHaGG [SAND '24]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is less homesick than $\mathcal{D}_2 = (f_2, G_2, P_2)$
  if $R_1(t) \leq R_2(t)$ for any t where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of $\mathcal{D}_1$ and $\mathcal{D}_2$ at time t.

- $\mathcal{D} = (f, G, P)$ is less homesick as graph growing (**LHaGG**)
  if $\mathcal{D}$ is less homesick than $\mathcal{D}' = (g, G, P)$ for any g satisfying that
  $\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} g(k)$ for any n,
  i.e., $\mathcal{D}$ and $\mathcal{D}'$ grows similarly, but $\mathcal{D}$ grows *faster.*

The faster a graph grows,
the smaller the return probability.

Theorems by LHaGG

The faster a graph grows,
the smaller the return probability.

Under the condition of LHaGG, we can prove the following sufficient conditions of recurrence/transience, respectively.

Thm. [Kumamoto, K., Shirai '24]
Suppose $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG. If

$$\sum_{n=1}^{\infty} \mathfrak{d}(n) p(n) = \infty$$

then $\mathcal{D}$ is recurrent at v, where $p(n) = \pi_n(v)$.

Thm. [Kumamoto, K., Shirai '24]
Suppose $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG. If

$$\sum_{n=1}^{\infty} \max\{\mathfrak{d}(n), \mathfrak{t}(n)\}\, p(n) < \infty$$

then $\mathcal{D}$ is transient at v, where $\mathfrak{t}(n)$ represents the mixing time.

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

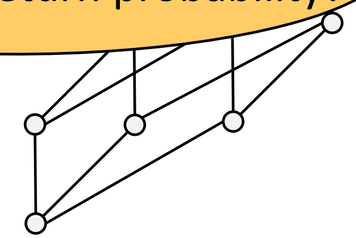
Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

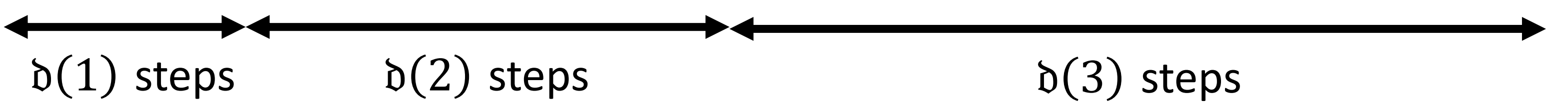
- $\mathfrak{d}(n) = 2^n$,
- $G(n)$ is a $\{0,1\}^n$ skeletone,
- $P(n)$ denotes the simple random walk,
  i.e., move to a neighbor w.p. $1/n$,

for $n = 1,2,\ldots$

The faster a graph grows,
the smaller the return probability?

Lem. [Kumamoto et al. 2024]
Random walk on $\{0,1\}^n$ is ***LHaGG***.





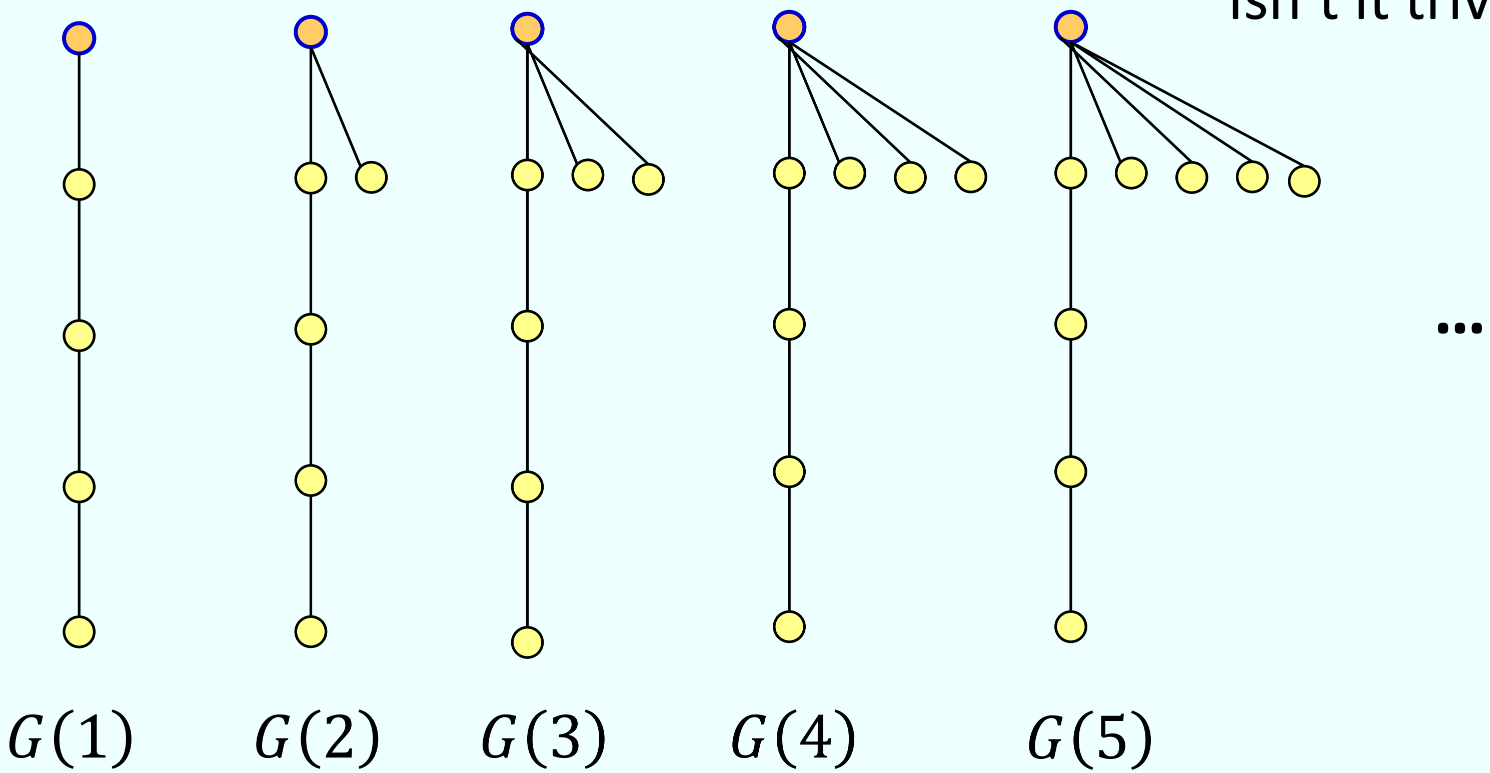
Thm. [Kumamoto et al. 2024]
If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

FAQ: Any example for ***not*** LHaGG?

The faster a graph grows, the smaller the return probability.

A (lazy) simple random walk on

Isn’t it trivial?



is ***not*** LHaGG.

Lazy RW on $\{0,1\}^n$ w/ increasing n is LHaGG

[SAND '24]

Proof.

The proof is a **monotone coupling**.

- Let $X_t \sim \mathcal{D}_f = (f, G, P)$ and $Y_t \sim \mathcal{D}_g = (g, G, P)$ where $\sum_{i=1}^{n} f(i) \geq \sum_{i=1}^{n} g(i)$,
  - i.e., the graph of $\mathcal{D}_g$ grows faster than that of $\mathcal{D}_f$.
- Let $|X_t|, |Y_t|$ denote the number of 1s in $X_t \in \{0,1\}^{n_t}$, $Y_t \in \{0,1\}^{m_t}$ where notice that $n_t \leq m_t$. Then,

$$\Pr[|X_{t+1}| - 1 = |X_t|] = \frac{1}{2}\frac{|X_t|}{n_t},\ \Pr[|X_{t+1}| = |X_t|] = \frac{1}{2},\ \Pr[|X_{t+1}| + 1 = |X_t|] = \frac{1}{2}\left(1 - \frac{|X_t|}{n_t}\right)$$
$$\Pr[|Y_{t+1}| - 1 = |Y_t|] = \frac{1}{2}\frac{|Y_t|}{m_t},\quad \Pr[|Y_{t+1}| = |Y_t|] = \frac{1}{2},\quad \Pr[|Y_{t+1}| + 1 = |Y_t|] = \frac{1}{2}\left(1 - \frac{|Y_t|}{m_t}\right)$$

- if $|X_t| < |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$
  - thanks to the self-loop w.p. $\frac{1}{2}$.
- If $|X_t| = |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$ since $n_t \leq m_t$.

Thus, $X_t = o$ if $Y_t = o$,
meaning that $\Pr[X_t = o] \geq \Pr[Y_t = o]$. □

It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.

5. まとめ

参入お待ちしてます.

まとめ

頂点が増えるグラフ上のRWの解析

- ✓ **“Cover time”**
  - ➢ 増えるクーポン収集問題
  - ➢ 期間$\mathfrak{d}$と未訪問数$\mathrm{E}[U_n]$の関係
- ✓ **再帰性**

今後の課題

- ✓ 正則グラフならもっと早く成長してもcoverできるか？
- ✓ Cover time、再帰性以外の指標
  - ➢ 時刻tの分布
  - ➢ “mixing time”
    - • Conductanceの研究は少しある [Dembo et al. 2017]
- ✓ 頂点が増える＋辺接続も変化する
  - ➢ 「適当な」条件

The end

Thank you for the attention.