

# The Recurrence/Transience of Random Walks on a Bounded Grid in an Increasing Dimension

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Shuma Kumamoto (Kyushu Univ.),

\*Shuji Kijima (Shiga Univ.),

Tomoyuki Shirai (Kyushu Univ.)

# Plan of talk

## 1. Introduction

- $\mathbb{Z}^3$
- RWoGG
- Tree

## 2. Related work

- Exploration

## 3. Previous work

- LHaGG
- $\{0,1\}^n$  proof
- Extension to  $\{0,1, \dots, N\}^n$

## 4. Main result

- Weakly LHaGG
- Recurrence
- Transience
- pausing coupling

## 5. Concluding remarks

## Plan of talk $\geq 49$ min.

1. Introduction ( $\geq 9$  min.)
  - $\mathbb{Z}^3$
  - RWoGG
  - Tree
2. Related work ( $\geq 6$  min.)
  - Exploration
3. Previous work ( $\geq 8$  min.)
  - LHaGG
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  - Extension to  $\{0,1, \dots, N\}^n$
4. Main result ( $\geq 25$  min.)
  - Weakly LHaGG
  - Recurrence
  - Transience
  - pausing coupling
5. Concluding remarks (1 min.)

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### 1. Introduction ( ~~$\geq 9$ min.~~ 6 min.)

- $\mathbb{Z}^3$
- RWoGG
- ~~Tree~~

### 2. Related work ( ~~$\geq 6$ min.~~ 3 min.)

- ~~Exploration~~

### 3. Previous work ( ~~$\geq 8$ min.~~)

- LHaGG
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- Extension to  $\{0,1, \dots, N\}^n$

### 4. Main result ( ~~$\geq 25$ min.~~ 7 min.)

- Weakly LHaGG
- ~~Recurrence~~
- ~~Transience~~
- pausing coupling

### 5. Concluding remarks (1 min.)

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Shuji Kijima

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# 1. Introduction w/ examples

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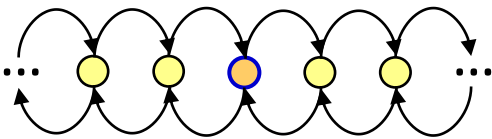
## Recurrence/Transience of Random walks on *infinite* graphs

A random walk on an infinite graph is **recurrent** at vertex  $v$  if it visits  $v$  **infinitely many times**, i.e.,

$$\sum_{t=0}^{\infty} \Pr[X_t = v] = \infty$$

holds, otherwise it is said to be **transient**.

For instance,



RW on  $\mathbb{Z}$  is **recurrent** at  $0$ ,

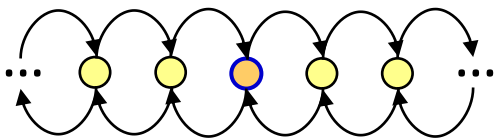
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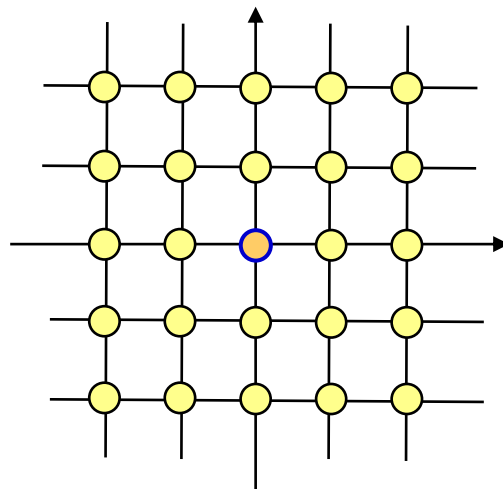
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RW on  $\mathbb{Z}$  is **recurrent** at  $o$ ,



RW on  $\mathbb{Z}^2$  is **recurrent** at  $o$ ,

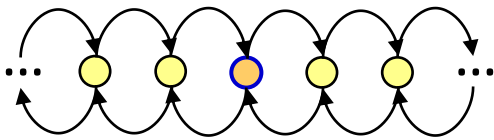
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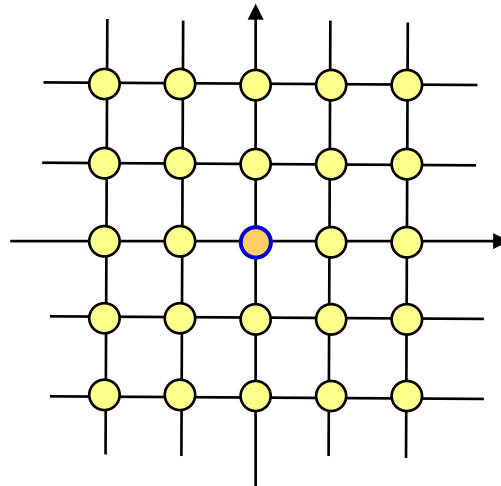
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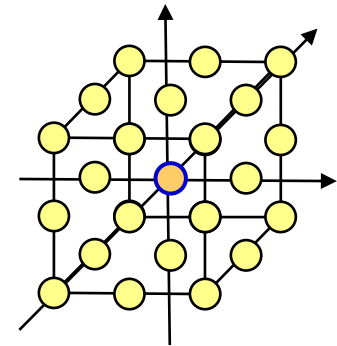
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RW on  $\mathbb{Z}$  is **recurrent** at  $o$ ,



RW on  $\mathbb{Z}^2$  is **recurrent** at  $o$ ,



RW on  $\mathbb{Z}^3$  is **transient** at  $o$ ,

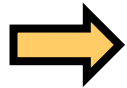


## Example 1. Random walk in a growing region of $\mathbb{Z}^3$

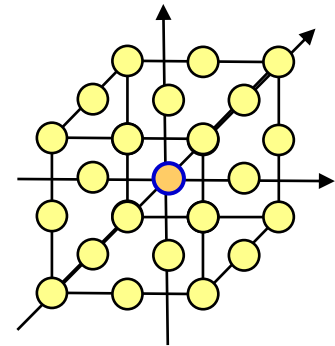
- ✓ Random walk on  $\mathbb{Z}^3$  is **transient** at  $o$ .
- ✓ Random walk on  $\{-n, \dots, n\}^3$  is **recurrent** at  $o$ .

Q. Is a random walk on  $\{-n, \dots, n\}^3$  **recurrent** or **transient** if  $n$  **increases** as time go on?

A. It depends on the increasing speed.



Find the phase transition point regarding the growing speed.



RW on  $\mathbb{Z}^3$  is **transient** at  $o$ ,

## Model: Random Walk on a Growing Graph (RWoGG)

□ Growing graph is a sequence of static graphs

[K, Shimizu, Shiraga '21]

$$\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots$$

where each  $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$  is a static simple graph.

We assume  $\mathcal{V}_t \subseteq \mathcal{V}_{t+1}$ , for convenience.

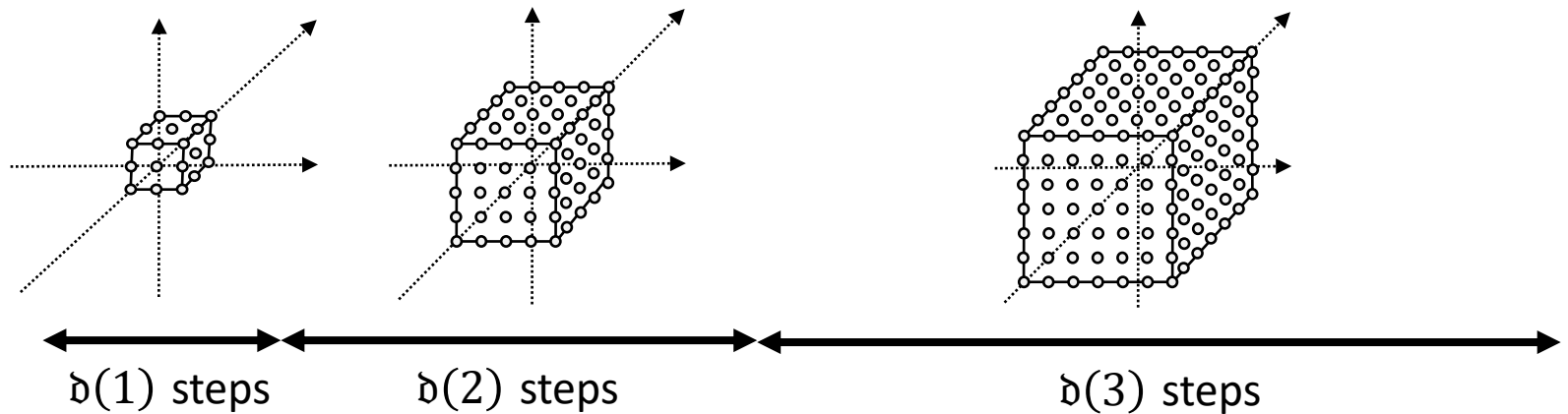
Furthermore,  $\mathcal{E}_t \subseteq \mathcal{E}_{t+1}$  holds in this talk.



## Example 1. Random walk in a growing region of $\mathbb{Z}^d$

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

- $\mathfrak{d}(n) = n^2$ ,
- $G(n)$  is a grid graph  $\{-n, \dots, n\}^3$ ,
- $P(n)$  denotes the simple random walk w/ reflection bound,  
i.e., move to a neighbor w.p.  $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$  unless boundary,  
for  $n = 1, 2, \dots$



Thm. [Dembo et al. 2014, Kumamoto et al. 2024]

If  $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{n^d} = \infty$  then recurrent, otherwise transient.

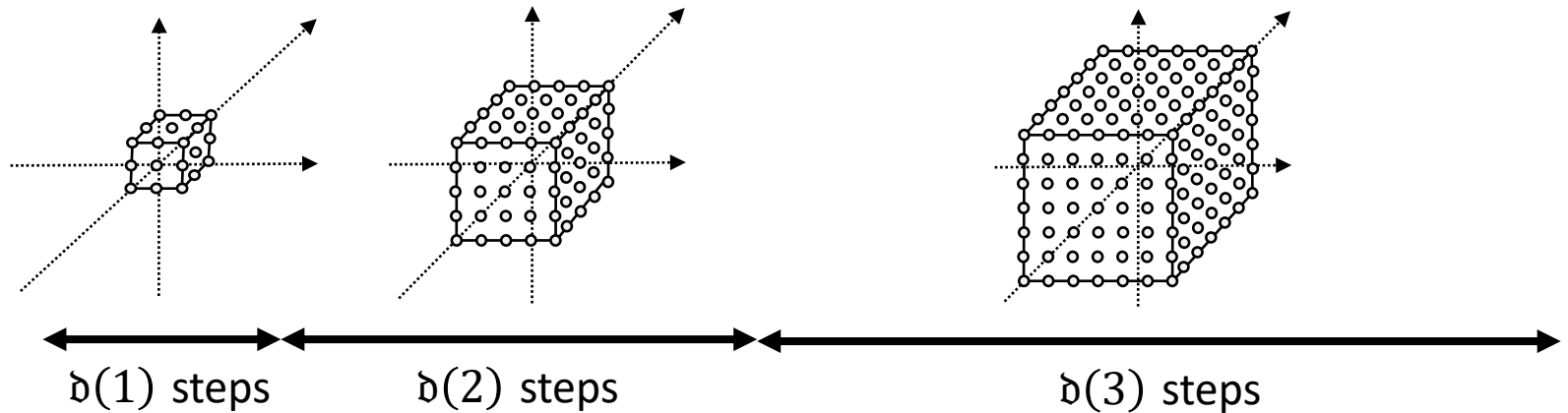
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**Recurrent**

$$\text{since } \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$



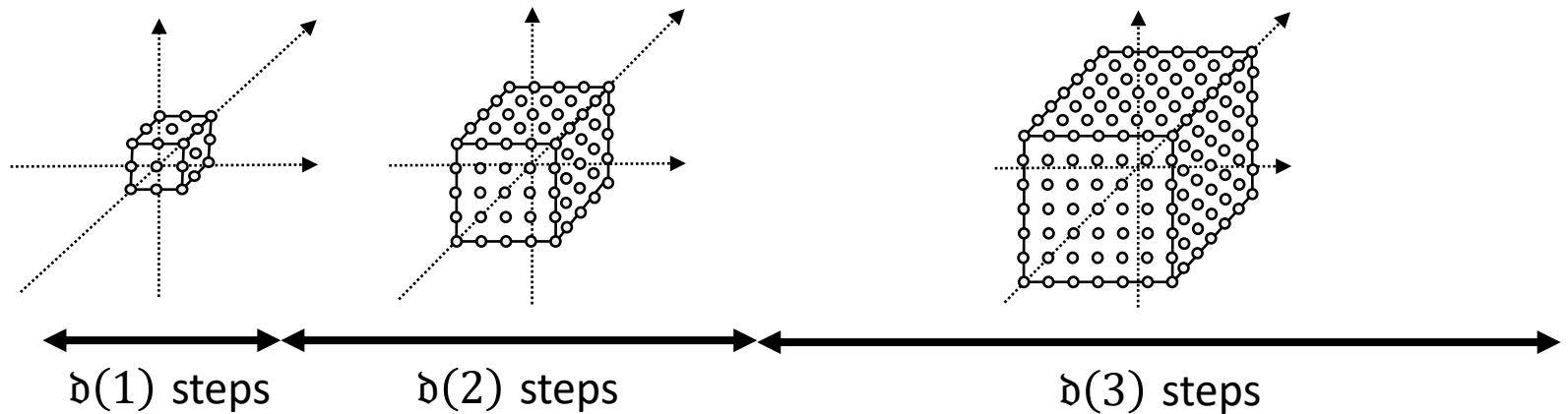
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If  $\sum_{n=1}^{\infty} \frac{\delta(n)}{n^d} = \infty$  then recurrent, otherwise transient.

## Example 1. Random walk in a growing region of $\mathbb{Z}^d$

Let  $\mathcal{D} = (\delta, G, P)$  be a RWoGG where

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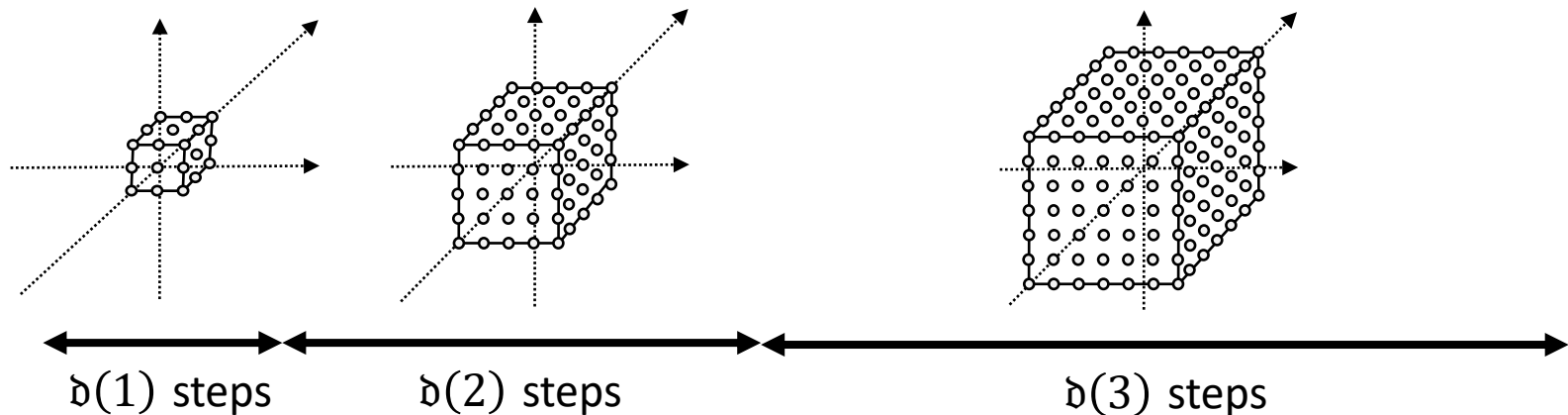
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**Transient**

$$\text{since } \sum_{n=1}^{\infty} \frac{n^{1.999}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{0.999}} < 1000.$$

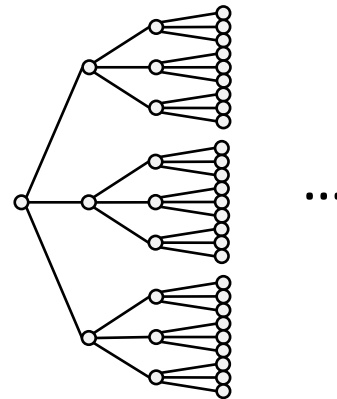


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## Example 2. RW on an infinite $k$ -ary tree

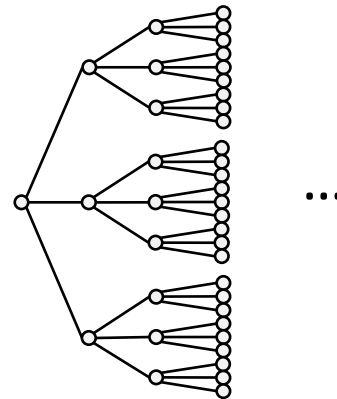
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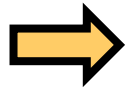


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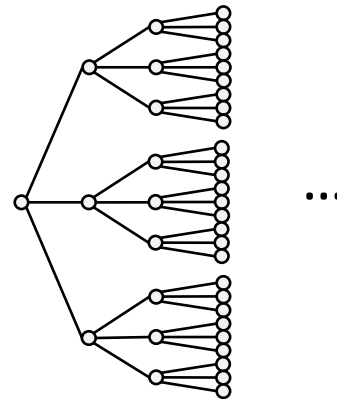
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A. It depends on the increasing speed.



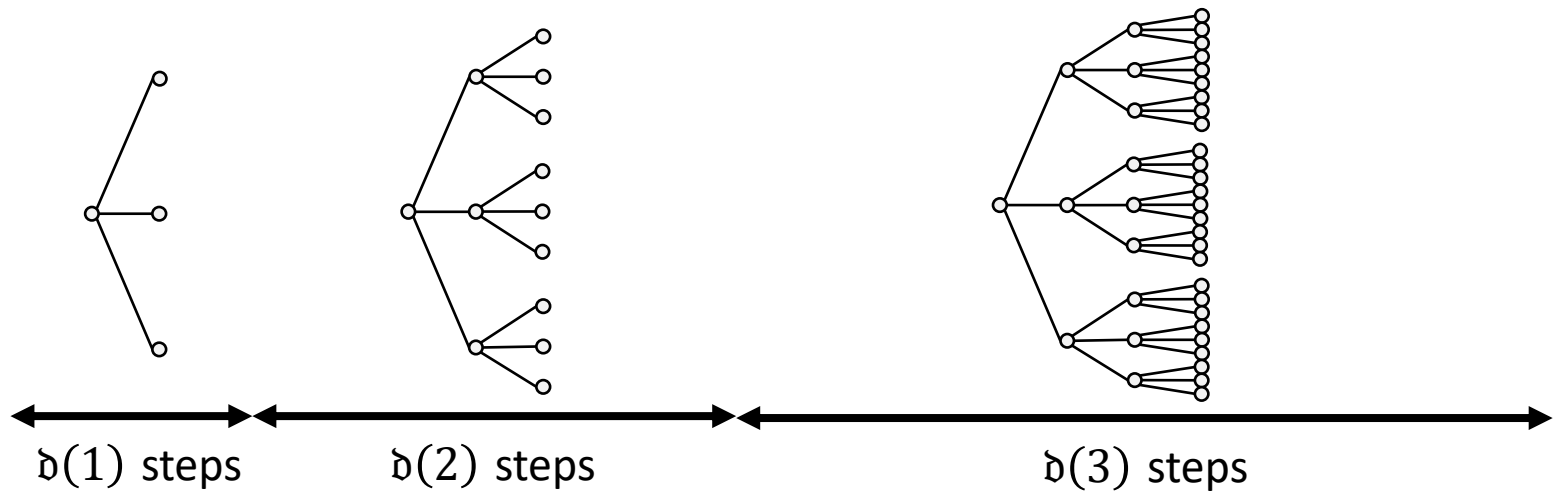
Find the phase transition point regarding the growing speed.



## Example 2. Random walk on a growing $k$ -ary tree

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

- $\mathfrak{d}(n) = 3^n$ ,
- $G(n)$  is a **3**-ary tree of height  $n$ ,
- $P(n)$  denotes the simple random walk w/ reflection bound, i.e., move to a neighbor w.p.  $1/4$  unless the root or a leaf, for  $n = 1, 2, \dots$



Thm. [Huang 2019, Kumamoto et al. 2024]

If  $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{k^n} = \infty$  then recurrent, otherwise transient.

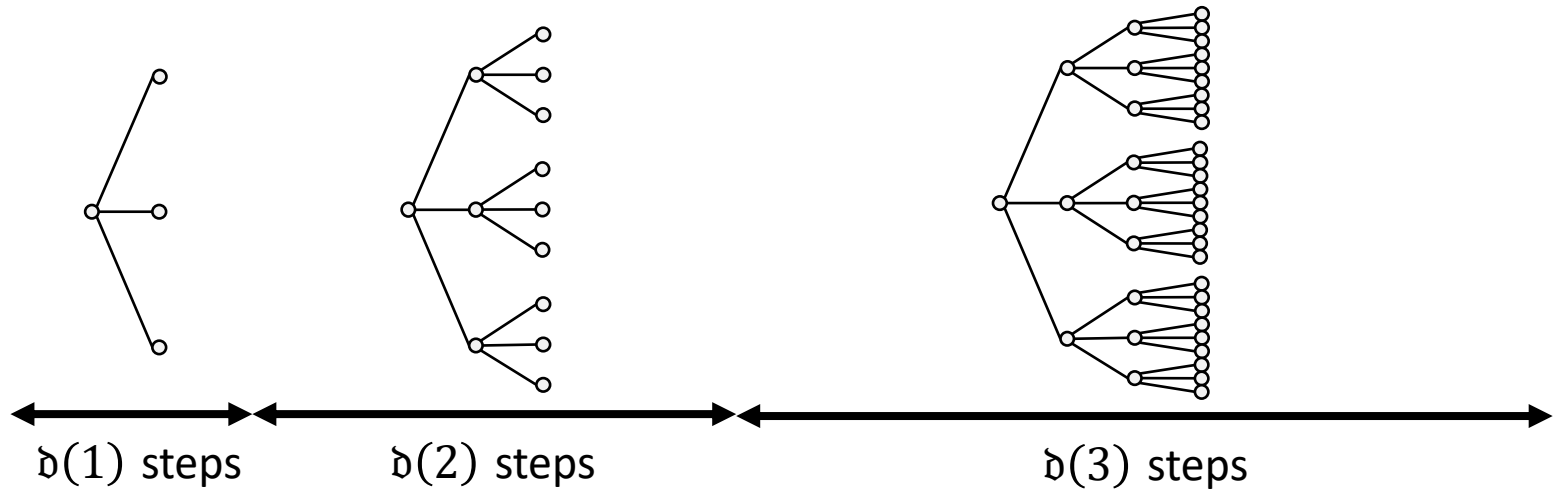
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**Recurrent**

$$\text{since } \sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1 = \infty.$$



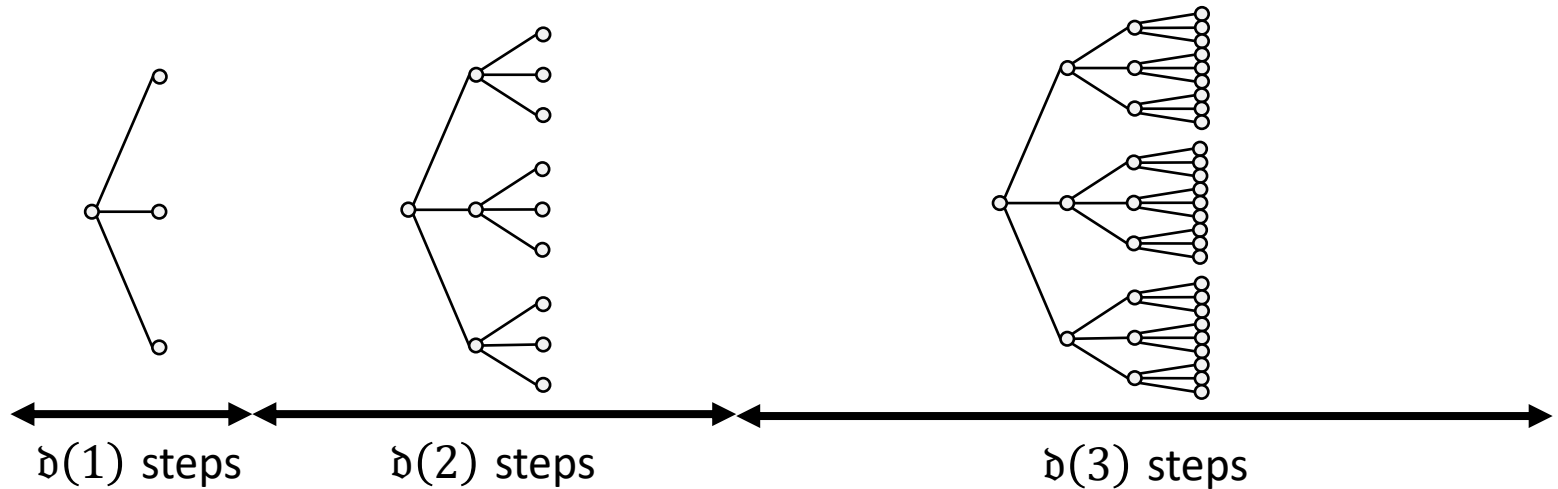
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Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

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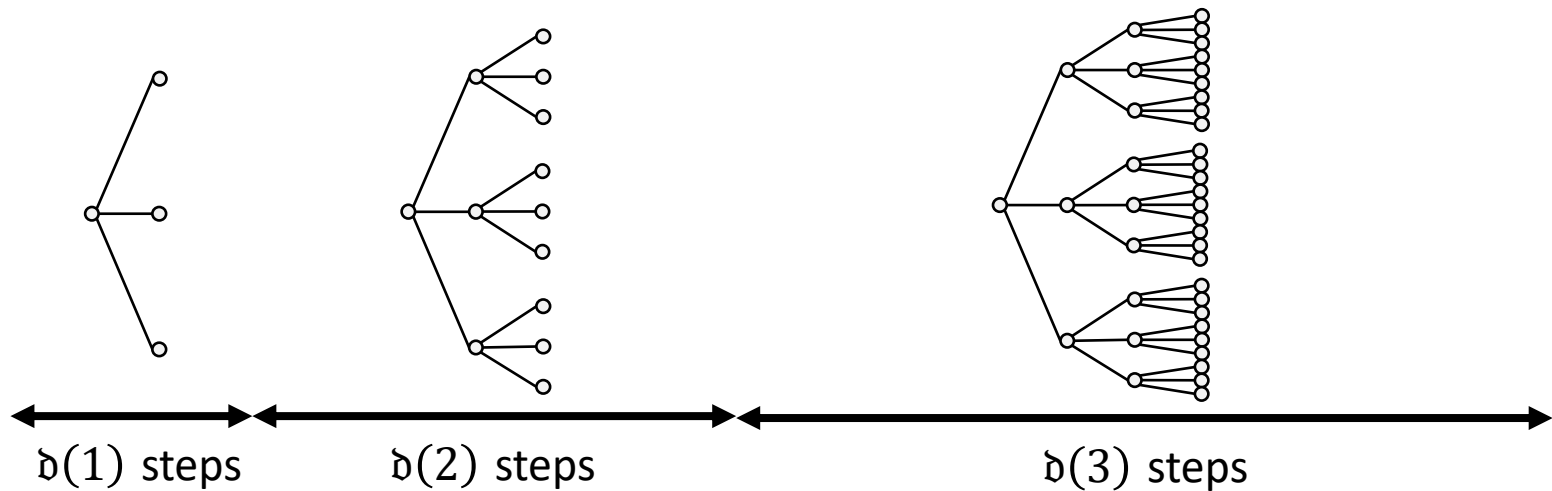
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**Transient**

since  $\sum_{n=1}^{\infty} \frac{2.999999^n}{3^n} < 1,000,000.$



Thm. [Huang 2019, Kumamoto et al. 2024]

If  $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{k^n} = \infty$  then recurrent, otherwise transient.



## 2. Related work

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About analysis of algorithms in dynamic environment

## Related work (1/2): Random walks on dynamic graphs

### □ Graph search by RW --- related to cover time

- Copper and Frieze (2003): Crawling on simple models of web graphs.
- Avin, Koucky and Lotker (2008): a bad example for hitting-time (cover time as well) w/  $\Omega(2^n)$  for the number of vertices  $n$ .
- Denysyuk and Rodrigues (2014): cover time under some fairness condition.
- Lamprou, Martin and Spirakis (2018): edge-uniform stochastically graphs.
- Sauerwald and Zanetti (2019):  $O(n^2)$  cover time for  $d$ -regular graphs.
- K, Shimizu, Shiraga (2021): cover ratio of **RWoGG**

### □ Mixing time

- Saloff-Coste and Zuniga (2009,2011): mixing time for time-inhomogeneous Markov chains w/ an invariant stationary distribution.
- Dembo, Huang and Zheng (2019) analyzed the conductance of growing subregion of  $\mathbb{Z}^d$ .
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### □ Recurrence/transience

... Continued



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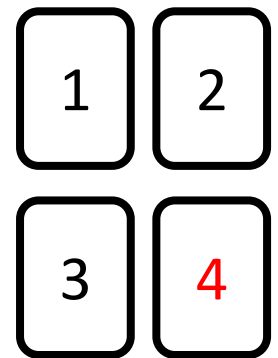
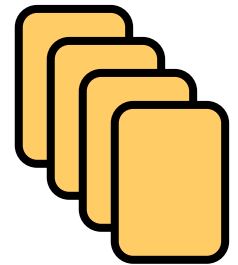
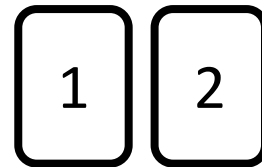
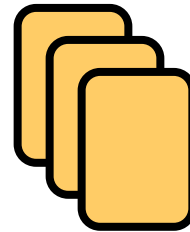
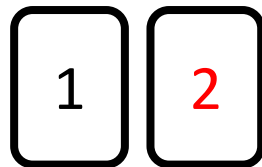
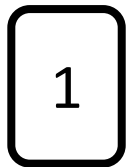
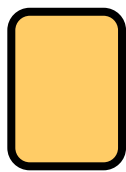
### □ Recurrence/transience

... Continued

# Collecting an **increasing** number of coupons [K, Shimizu, Shiraga '21]

Day	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9
# types	1	2	2	3	3	3	4	4	4
$\Pr[X_t = k]$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

1<sup>st</sup> period      2<sup>nd</sup> period      3<sup>rd</sup> period      4<sup>th</sup> period

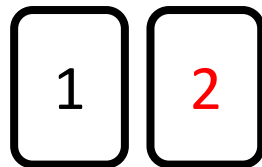
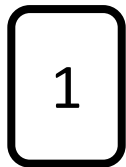
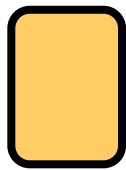


Q. How many types are collected in the end of  $n^{\text{th}}$  period?

# Collecting an **increasing** number of coupons [K, Shimizu, Shiraga '21]

Day	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9
# types	1	2	2	3					
$\Pr[X_t = k]$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$					

1<sup>st</sup> period      2<sup>nd</sup> period      3<sup>rd</sup>



1.  $O(\log n)$
2.  $O(\sqrt{n})$
3.  $\frac{n}{2}$
4.  $.99n$

Q. How many types are collected in the end of  $n^{\text{th}}$  period?

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1.  $O(\log n)$   
 2.  $O(\sqrt{n})$   
 3.  $\frac{n}{2}$   
 4.  $.99n$   
**5. at least  $n - 1$  in expectation**

Q. How many types are collected in the end of  $n^{\text{th}}$  period?

## Collecting an increasing number of coupons

Draw a coupon everyday

$\delta(n)$ : #days of the  $n^{\text{th}}$  period

$U_n$ : #items uncollected

in the end of  $n^{\text{th}}$  period

[K, Shimizu, Shiraga '21]

Prop.

$$\text{If } \delta(n) = n \text{ then } E[U_n] < \frac{1}{e-1}.$$

Proof.

$$\checkmark \quad \mathcal{E}_{i,n} := \begin{cases} 1 & \text{(item } i \text{ is uncollected in the end of the } n^{\text{th}} \text{ period)} \\ 0 & \text{(item } i \text{ is collected by the end of the } n^{\text{th}} \text{ period)} \end{cases}$$

for  $i = 1, 2, \dots, n$ .

$$\checkmark \quad U_n = \sum_{i=1}^n \mathcal{E}_{i,n}$$

✓ Prob. that item  $n$  is uncollected in the end of the  $n^{\text{th}}$  period:

$$\Pr[\mathcal{E}_{n,n} = 1] = \left(1 - \frac{1}{n}\right)^n < e^{-1}$$

✓ Prob. that item  $i$  ( $i \leq n$ ) is uncollected in the end of the  $n^{\text{th}}$  period:

$$\Pr[\mathcal{E}_{i,n} = 1] = \left(1 - \frac{1}{i}\right)^i \left(1 - \frac{1}{i+1}\right)^{i+1} \dots \left(1 - \frac{1}{n}\right)^n < \left(\frac{1}{e}\right)^{n+1-i}$$

$$\checkmark \quad E[U_n] = \sum_{i=1}^n \Pr[\mathcal{E}_{i,n}] < \sum_{i=1}^n \left(\frac{1}{e}\right)^{n+1-i} = \frac{1}{e} + \frac{1}{e^2} + \dots + \frac{1}{e^n} < \frac{\frac{1}{e}}{1 - \frac{1}{e}} = \frac{1}{e-1} < 0.582.$$

## RWoGG ( $\delta, G, P$ )

Coupon collector is often regarded as a RW on the complete graph, and we can extend the arguments to RWoGG for general graphs.

### Thm. (general upper bound)

If  $\delta(i) \geq ct_{\text{hit}}(i)$  ( $c \geq 1$ ) then  $E[U] = O(1)$ .

Particularly, if  $\frac{\delta(i)}{t_{\text{hit}}(i)} \xrightarrow{i \rightarrow \infty} \infty$  then  $E[U_n] \xrightarrow{n \rightarrow \infty} 0$ .

### Thm. (upper bound for lazy and reversible walk)

Suppose  $P^{(i)}$  is lazy and reversible.

If  $\frac{t_{\text{hit}}(i)}{t_{\text{mix}}(i)} \geq \frac{i^\gamma}{c}$  and  $\delta(i) \geq \frac{3ct_{\text{hit}}(i)}{i^\gamma}$  ( $c > 0$ ) then  $E[U_n] \leq \frac{8n^\gamma}{c} + 32$ .

S. Kijima, N. Shimizu, T. Shiraga, How many vertices does a random walk miss in a network with moderately increasing the number of vertices?, in Proc. SODA 2021, 106–122.

## Related work (2/2): recurrence/transience of RW

- Much work about the recurrence/transience on growing graphs exist in the context of self-interacting random walks including reinforced random walks, excited random walks, etc. since 1990s, or before.
- Dembo, Huang and Sidoravicius (2014× 2): recurrent  $\Leftrightarrow \sum_{t=0}^{\infty} \pi_t(0) = \infty$  for growing subregion of  $\mathbb{Z}^d$  (fixed  $d$ ), by conductance argument.
  - See also Huang and Kumagai (2016), Dembo, Huang, Morris and Peres (2017), Dembo, Huang and Zheng (2019), etc. about heat kernel, evolving set arguments.
- Amir, Benjamini, Gurel-Gurevich and Kozma (2015): random walk on growing tree. (random walk in changing environment).
- Huang (2017): growing graph w/ *uniformly bounded degrees*.
- Kumamoto, K. and Shirai (2024):  $k$ -ary tree,  $\{0,1\}^n$  w/ an increasing  $n$  under **RWoGG** model by coupling.
- This work (2024):  $\{0,1, \dots, N\}^n$  (fixed  $N$ , increasing  $n$ ) by pausing coupling.



### 3. Our previous work [SAND '24]

---

About the recurrence/transience of RWoGG,  
for an introduction of the basic technique and its issue.

S. Kumamoto, S. Kijima and T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, Proc. SAND 2024, 17:1-17:17

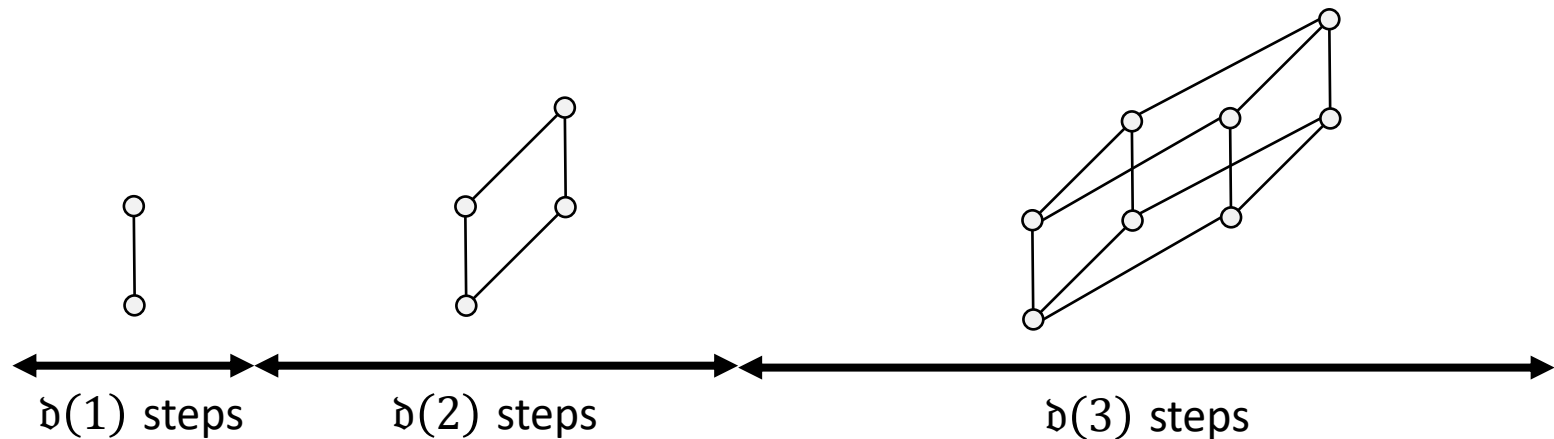


### Example 3. Random walk on $\{0,1\}^n$ w/ an increasing $n$

[SAND '24]

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

- $\mathfrak{d}(n) = 2^n$ ,
  - $G(n)$  is a  $\{0,1\}^n$  skeleton,
  - $P(n)$  denotes the simple random walk,  
i.e., move to a neighbor w.p.  $1/n$ ,
- for  $n = 1, 2, \dots$



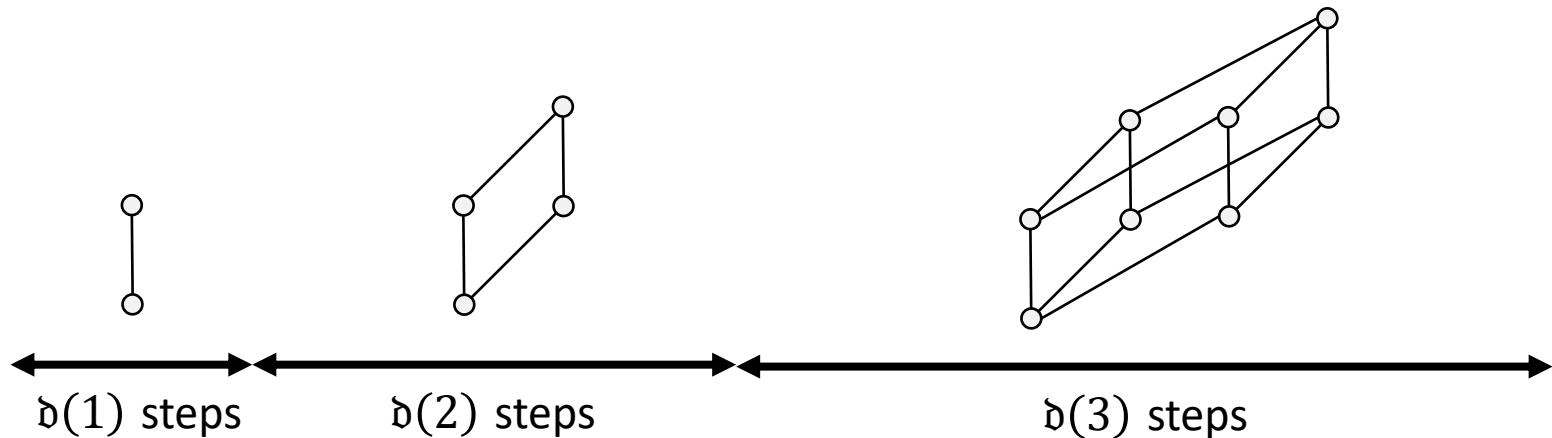
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**Recurrent**  
since  $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$ .



Thm. [Kumamoto et al. 2024]

If  $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$  then recurrent, otherwise transient.

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[SAND '24]

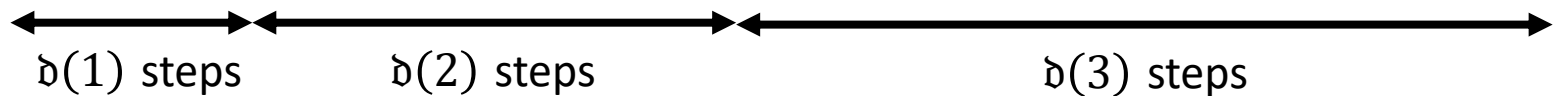
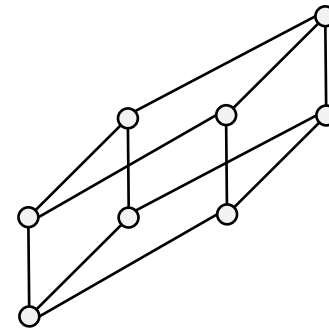
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Lem. [Kumamoto et al. 2024]

Random walk on  $\{0,1\}^n$  is **LHaGG**.



Thm. [Kumamoto et al. 2024]

If  $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$  then recurrent, otherwise transient.

## LHaGG [SAND '24]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$  is **less homesick** than  $\mathcal{D}_2 = (f_2, G_2, P_2)$   
if  $R_1(t) \leq R_2(t)$  for any  $t$  where  $R_1(t)$  and  $R_2(t)$  respectively denote the return probabilities of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  at time  $t$ .
- $\mathcal{D} = (f, G, P)$  is **less homesick as graph growing (LHaGG)**  
if  $\mathcal{D}$  is less homesick than  $\mathcal{D}' = (g, G, P)$  for any  $g$  satisfying that  
 $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$  for any  $n$ ,  
i.e.,  $\mathcal{D}$  and  $\mathcal{D}'$  grows similarly, but  $\mathcal{D}$  grows *faster*.

The faster a graph grows,  
the smaller the return probability.

## Theorems by LHaGG

The faster a graph grows,  
the smaller the return probability.

Under the condition of LHaGG, we can prove the following sufficient conditions of recurrence/transience, respectively.

Thm. [\[Kumamoto, K., Shirai '24\]](#)

Suppose  $\mathcal{D} = (\mathfrak{d}, G, P)$  is LHaGG. If

$$\sum_{n=1}^{\infty} \mathfrak{d}(n)p(n) = \infty$$

then  $\mathcal{D}$  is **recurrent** at  $v$ , where  $p(n) = \pi_n(v)$ .

Thm. [\[Kumamoto, K., Shirai '24\]](#)

Suppose  $\mathcal{D} = (\mathfrak{d}, G, P)$  is LHaGG. If

$$\sum_{n=1}^{\infty} \max\{\mathfrak{d}(n), t(n)\} p(n) < \infty$$

then  $\mathcal{D}$  is **transient** at  $v$ , where  $t(n)$  represents the mixing time.

### Example 3. Random walk on $\{0,1\}^n$ w/ an increasing $n$

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

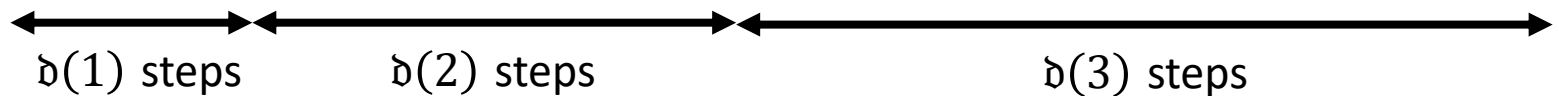
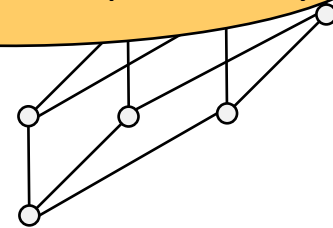
- $\mathfrak{d}(n) = 2^n$ ,
- $G(n)$  is a  $\{0,1\}^n$  skeleton,
- $P(n)$  denotes the simple random walk,  
i.e., move to a neighbor w.p.  $1/n$ ,

for  $n = 1, 2, \dots$

Lem. [Kumamoto et al. 2024]

Random walk on  $\{0,1\}^n$  is **LHaGG**.

The faster a graph grows,  
the smaller the return probability?



Thm. [Kumamoto et al. 2024]

If  $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$  then recurrent, otherwise transient.

FAQ: Any example for *not* LHaGG?

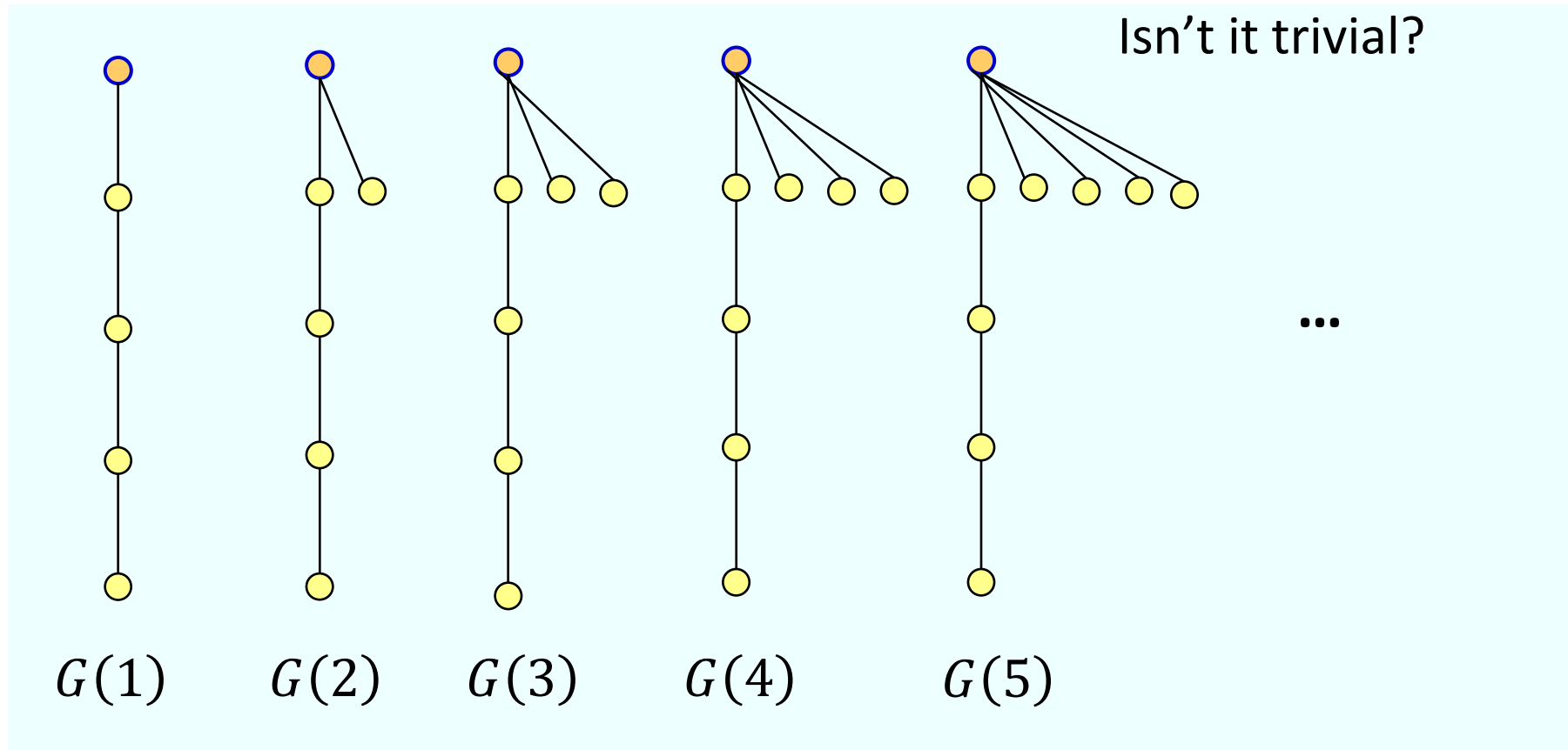
The faster a graph grows,  
the smaller the return probability.

Isn't it trivial?

## FAQ: Any example for *not* LHaGG?

A (lazy) simple random walk on

The faster a graph grows, the smaller the return probability.



is *not* LHaGG.



## Lazy RW on $\{0,1\}^n$ w/ increasing $n$ is LHaGG

[SAND '24]

Proof.

The proof is a **monotone coupling**.

- Let  $X_t \sim \mathcal{D}_f = (f, G, P)$  and  $Y_t \sim \mathcal{D}_g = (g, G, P)$  where  $\sum_{i=1}^n f(i) \geq \sum_{i=1}^n g(i)$ ,
  - i.e., the graph of  $\mathcal{D}_g$  grows faster than that of  $\mathcal{D}_f$ .
- Let  $|X_t|, |Y_t|$  denote the number of 1s in  $X_t \in \{0,1\}^{n_t}, Y_t \in \{0,1\}^{m_t}$  where notice that  $n_t \leq m_t$ . Then,

$$\Pr[|X_{t+1}| - 1 = |X_t|] = \frac{1}{2} \frac{|X_t|}{n_t}, \quad \Pr[|X_{t+1}| = |X_t|] = \frac{1}{2}, \quad \Pr[|X_{t+1}| + 1 = |X_t|] = \frac{1}{2} \left(1 - \frac{|X_t|}{n_t}\right)$$

$$\Pr[|Y_{t+1}| - 1 = |Y_t|] = \frac{1}{2} \frac{|Y_t|}{m_t}, \quad \Pr[|Y_{t+1}| = |Y_t|] = \frac{1}{2}, \quad \Pr[|Y_{t+1}| + 1 = |Y_t|] = \frac{1}{2} \left(1 - \frac{|Y_t|}{m_t}\right)$$

- if  $|X_t| < |Y_t|$  then we can couple so that  $|X_{t+1}| \leq |Y_{t+1}|$ 
  - thanks to the self-loop w.p.  $\frac{1}{2}$ .
- If  $|X_t| = |Y_t|$  then we can couple so that  $|X_{t+1}| \leq |Y_{t+1}|$  since  $n_t \leq m_t$ .

Thus,  $X_t = o$  if  $Y_t = o$ ,

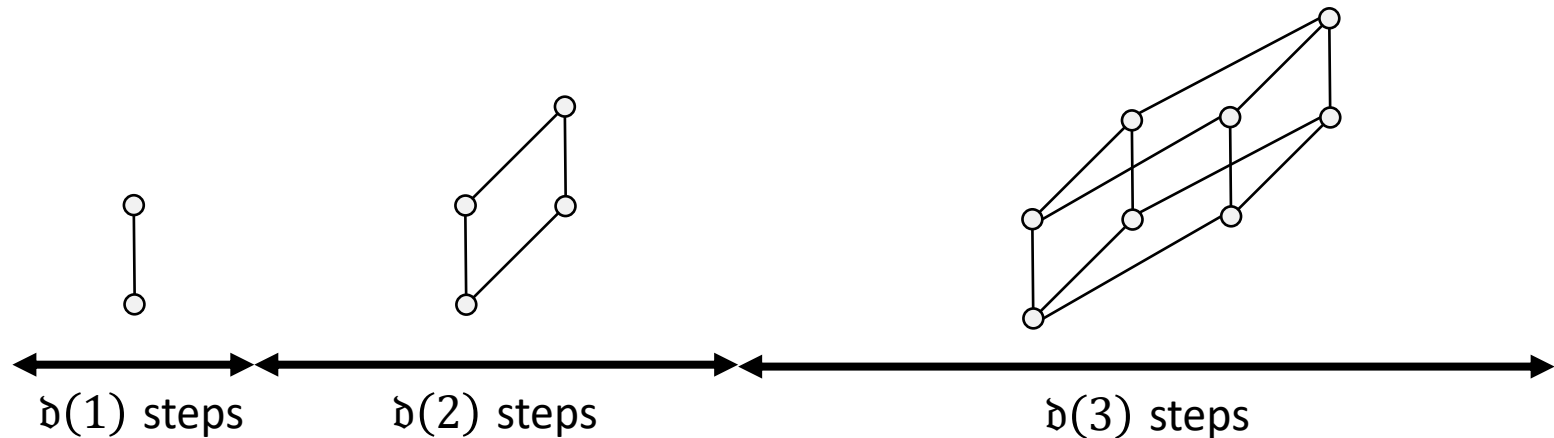
meaning that  $\Pr[X_t = o] \geq \Pr[Y_t = o]$ . □

It looks a very simple exercise if you are familiar with **coupling**, but  $n_t \neq m_t$  makes some trouble, in general.

### Example 3. Random walk on $\{0,1\}^n$ w/ an increasing $n$

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

- $\mathfrak{d}(n) = 2^n$ ,
  - $G(n)$  is a  $\{0,1\}^n$  skeleton,
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i.e., move to a neighbor w.p.  $1/n$ ,
- for  $n = 1, 2, \dots$



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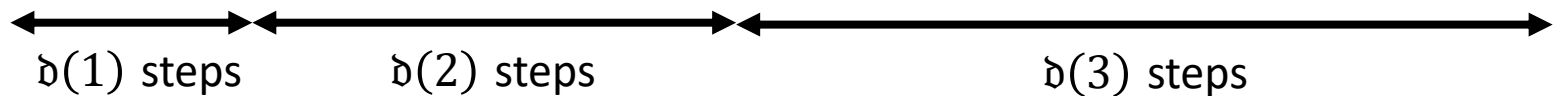
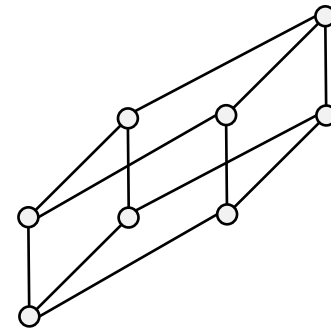
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**Recurrent**  
since  $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$ .

Lem. [Kumamoto et al. 2024]

Random walk on  $\{0,1\}^n$  is **LHaGG**.

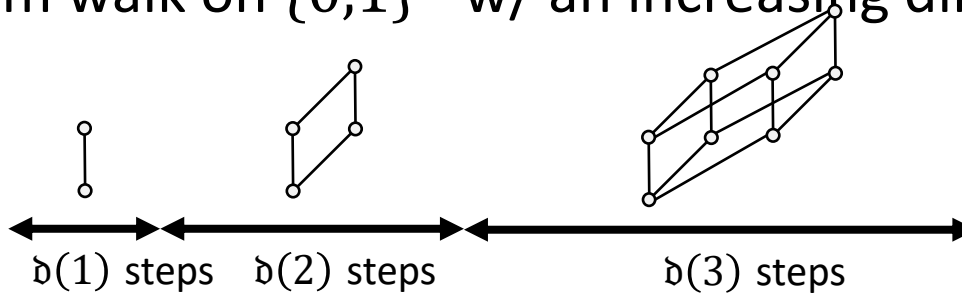


Thm. [Kumamoto et al. 2024]

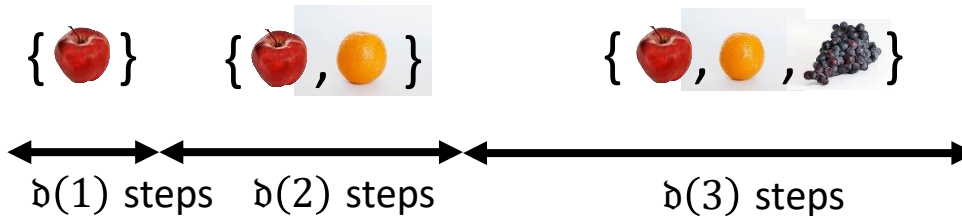
If  $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$  then recurrent, otherwise transient.

## Three representations (or “applications”?) of $\{0,1\}^n$

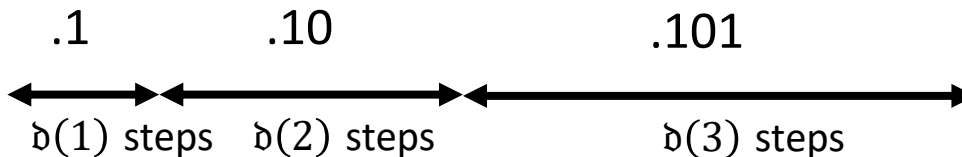
- Random walk on  $\{0,1\}^n$  w/ an increasing dimensions



- Random pick/drop items w/ an increasing number of items

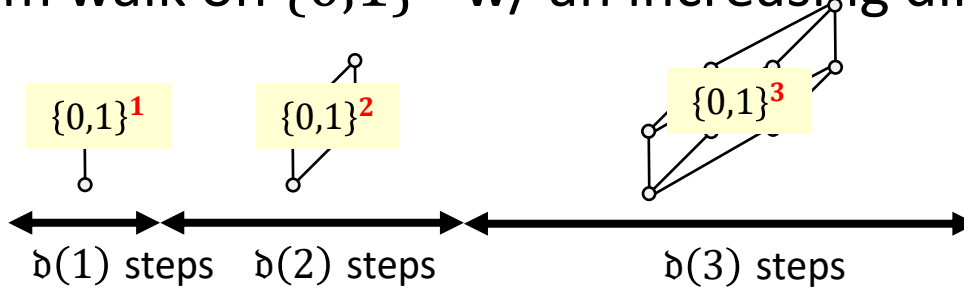


- Random bit flip of binary w/ an increasing bit length

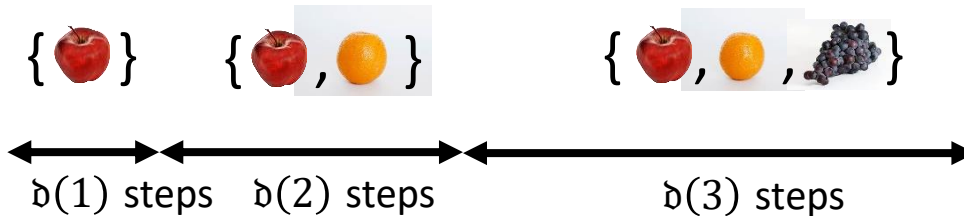


## Three representations (or “applications”?) of $\{0,1\}^n$

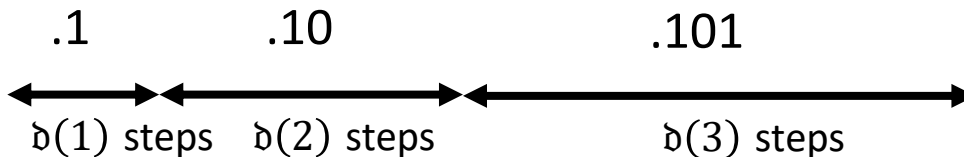
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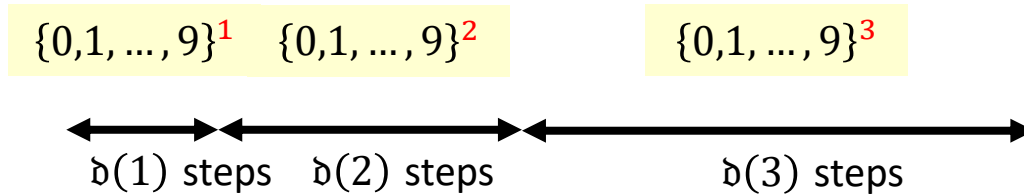


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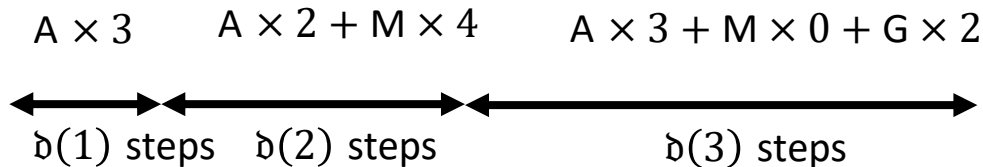


## Extension from $\{0,1\}^n$ to $\{0,1,\dots,9\}^n$

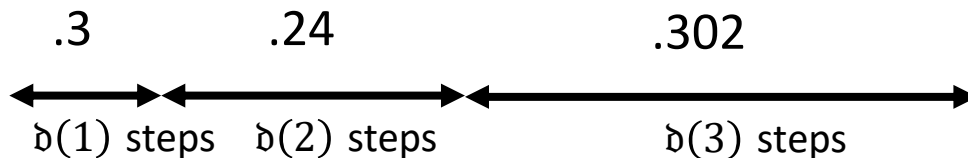
- Random walk on  $\{0,1,\dots,9\}^n$  w/ an increasing  $n$



- Random buy/sell stocks w/ an increasing #brands



- Random up/down digits w/ an increasing digit length

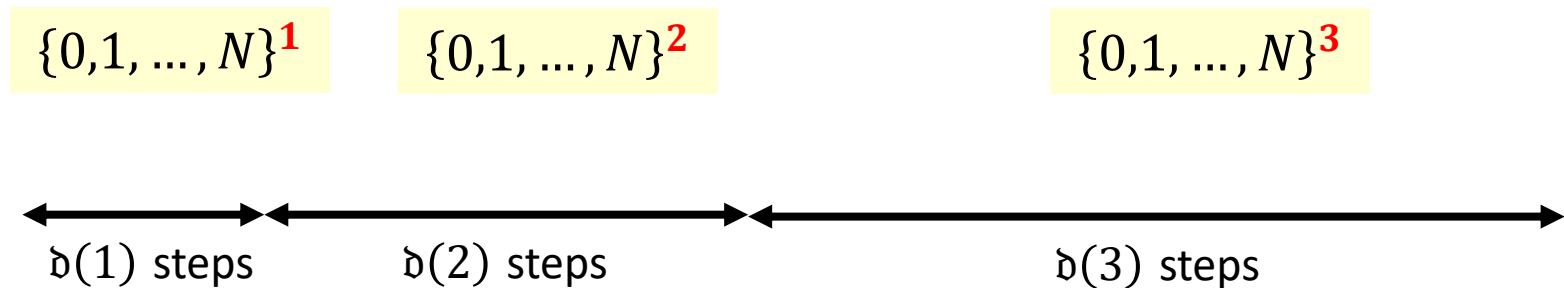


Target. Random walk on  $\{0,1, \dots, N\}^n$  w/ an increasing  $n$

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

w/ a fixed  $N$ .

- $\mathfrak{d}(n) = N^n$ ,
- $G(n)$  is a  $\{0,1, \dots, N\}^n$  skeleton,
- $P(n)$  denotes the lazy simple random walk w/ reflection bound, i.e., move to a neighbor w.p.  $1/4n$ , unless boundary for  $n = 1, 2, \dots$



Q.

Is random walk on  $\{0,1, \dots, N\}^n$  LHaGG?

A. We can't prove it.



## 4. Main Result

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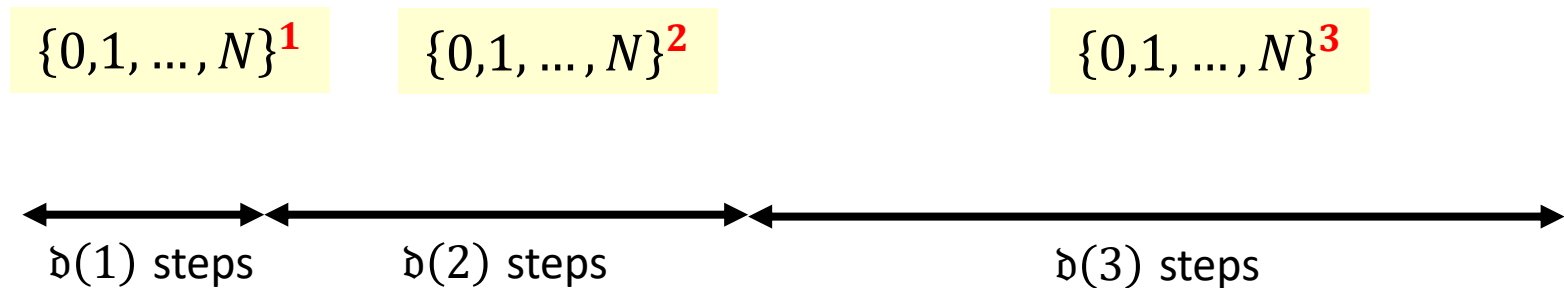


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Let  $\mathcal{D} = (\delta, G, P)$  be a RWoGG where

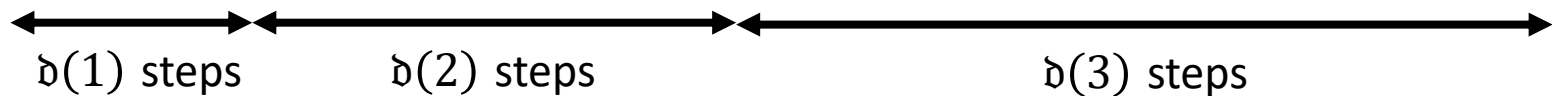
w/ a fixed  $N$ .

- $\delta(n) = N^n$ ,
- $G(n)$  is a  $\{0,1, \dots, N\}^n$  skeleton,
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Lem. 7.

Random walk on  $\{0,1, \dots, N\}^n$  is **weakly LHaGG**.

$N\}^3$



Q.

Is random walk on  $\{0,1, \dots, N\}^n$  LHaGG?

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Lem. 7.

Random walk on  $\{0,1, \dots, N\}^n$  is **weakly LHaGG**.

$\mathfrak{d}\}^3$

Thm. 6. If  $\mathcal{D} = (\mathfrak{d}, G, P)$  satisfies

$$\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{(2N)^n} = \infty$$

then  $o$  is recurrent, otherwise  $o$  is transient.

) steps

## Recall: LHaGG [SAND '24]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$  is **less homesick** than  $\mathcal{D}_2 = (f_2, G_2, P_2)$   
if  $R_1(t) \leq R_2(t)$  for any  $t$  where  $R_1(t)$  and  $R_2(t)$  respectively denote the return probabilities of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  at time  $t$ .
- $\mathcal{D} = (f, G, P)$  is **less homesick as graph growing (LHaGG)**  
if  $\mathcal{D}$  is less homesick than  $\mathcal{D}' = (g, G, P)$  for any  $g$  satisfying that  
 $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$  for any  $n$ ,  
i.e.,  $\mathcal{D}$  and  $\mathcal{D}'$  grows similarly, but  $\mathcal{D}$  grows *faster*.

The faster a graph grows,  
the smaller the **return probability**.

## Recall: LHaGG

We replace the condition about the return prob. with

Defs.

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 if  $R_1(t) \leq R_2(t)$  for any  $t$  where  $R_1(t)$  and  $R_2(t)$  respectively denote  
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 $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$  for any  $n$ ,  
 i.e.,  $\mathcal{D}$  and  $\mathcal{D}'$  grows similarly, but  $\mathcal{D}$  grows *faster*.

The faster a graph grows,  
 the smaller the **return probability**.

## wLHaGG

We replace the condition about the return prob. with a condition of the **sum of return prob.**

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$  is **weakly less homesick** than  $\mathcal{D}_2 = (f_2, G_2, P_2)$  if  $\sum_{t=1}^T R_1(t) \leq \sum_{t=1}^T R_2(t)$  for any  $T$  where  $R_1(t)$  and  $R_2(t)$  respectively denote the return probabilities of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  at time  $t$ .
- $\mathcal{D} = (f, G, P)$  is **weakly less homesick as graph growing (wLHaGG)** if  $\mathcal{D}$  is weakly less homesick than  $\mathcal{D}' = (g, G, P)$  for any  $g$  satisfying that  $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$  for any  $n$ , i.e.,  $\mathcal{D}$  and  $\mathcal{D}'$  grows similarly, but  $\mathcal{D}$  grows *faster*.

The faster a graph grows,  
the smaller the **expected number of returns**.

= sum of return prob.

## General theorems

Condition 0. (**ergodic**). In  $\mathcal{D} = (\mathfrak{d}, G, P)$ , every transition matrix  $P(n)$  is ergodic.

Condition 1. (**mixing time**).  $\mathcal{D} = (\mathfrak{d}, G, P)$  satisfies

$$\sum_{k=1}^{\infty} \tau^*(k) p(k) < \infty$$

where  $p(k) = \pi_k(o)$  and  $\tau^*(k) = t_{\text{mix}}^k \left( \frac{p(k)}{4} \right)$ .

Mixing time is not very big.

E.g.,  $O\left(\frac{1}{\pi_k(o)} \frac{1}{k \log k}\right)$

Thm. 2. (Recurrence).

Suppose  $(\mathfrak{d}, G, P)$  satisfies Conditions 0 and 1.

If  $\sum_{k=1}^{\infty} \mathfrak{d}(k) p(k) = \infty$  then the initial vertex  $v$  is **recurrent**.

Thm. 4. (Transience).

Suppose  $(\mathfrak{d}, G, P)$  satisfies Conditions 0 and 1, and it is **wLHaGG**.

If  $\sum_{k=2}^{\infty} \mathfrak{d}(k) p(k-1) < \infty$  then the initial vertex  $v$  is **transient**.

Thm. 2. (Recurrence).

Suppose  $(\mathfrak{d}, G, P)$  satisfies Conditions 0 and 1.

If  $\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty$  then the initial vertex  $v$  is **recurrent**.

## Recurrence

Proof. Let  $X_t$  follow  $(\mathfrak{d}, G, P)$ , and let  $R(t) = \Pr[X_t = o]$ . We claim

$$\text{Lem. 3. } \sum_{t=1}^{T_n} R(t) \geq \frac{1}{2} \sum_{k=1}^n (\mathfrak{d}(k) - \tau^*(k))p(k)$$

Proof of Lem. 3.

- Notice that  $X_t$  follows  $P_n$  for  $t \in [T_{n-1}, T_{n-1} + \mathfrak{d}(n))$ .
- If  $\mathfrak{d}(n) > t_{\text{mix}}(\epsilon)$  then  $R(t) \geq \pi_n(o) - \epsilon$  for  $t \geq T_{n-1} + t_{\text{mix}}(\epsilon)$  where  $\pi_n$  is the stationary distribution of  $P_n$ .
- Thus,  $R(t) \geq \pi_n(o) - \frac{1}{2}p(n) = \frac{1}{2}p(n)$   
since  $\tau^*(k) = t_{\text{mix}}\left(\frac{1}{2}p(n)\right)$  and  $p(n) = \pi_n(o)$ .
- $\sum_{t=1}^{T_n} R(t) = \sum_{k=1}^n \sum_{s=1}^{\mathfrak{d}(k)} R(T_{n-1} + s) \geq \sum_{k=1}^n \sum_{s=\tau^*(n)}^{\mathfrak{d}(k)} R(T_{n-1} + s) \geq \sum_{k=1}^n \sum_{s=\tau^*(n)}^{\mathfrak{d}(k)} \frac{1}{2}p(n) = \frac{1}{2} \sum_{k=1}^n (\mathfrak{d}(k) - \tau^*(k))p(k)$

Once we obtain Lem. 3, Thm. 2 is easy:  $\sum_{t=1}^{\infty} R(t) = \infty$  holds

if  $\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty$  and  $\sum_{k=1}^{\infty} \tau^*(k)p(k) < \infty$ .

← Mixing time condition



## Transience

Suppose  $(\mathfrak{d}, G, P)$  satisfies Conditions 0 and 1, and it is **wLHaGG**.

If  $\sum_{k=2}^{\infty} \mathfrak{d}(k)p(k-1) < \infty$  then the initial vertex  $v$  is **transient**.

Proof. Let  $f(k) = \max\{\mathfrak{d}, \tau^*(k)\}$ .

By wLHaGG,  $\sum_{t=1}^{T_n} R_{\mathfrak{d}}(t) \leq \sum_{t=1}^{T_n} R_g(t)$ .

**Lem. 5.  $\sum_{t=1}^{T_n} R_g(t) \leq g(1) + \frac{3}{2} \sum_{k=2}^n g(k)p(k-1)$**

Proof of Lem. 5.

Let  $f(k) = \begin{cases} g(k) & k \leq n-1 \\ \infty & k = n. \end{cases}$  Then,  $\sum_{k=1}^m g(k) \leq \sum_{k=1}^m f(k)$  for any  $m$ .

Let  $X_t \sim \mathcal{D}_g = (g, G, P)$  and  $Y_t \sim \mathcal{D}_f = (f, G, P)$ .

- Notice that  $Y_t$  follows  $P_{n-1}$  for  $t \geq T_{n-2}$ .
- By wLHaGG,  $\sum_{t=1}^T \Pr[X_t = o] \leq \sum_{t=1}^T \Pr[Y_t = o]$  for any  $T$ .
- $R_f(t) \leq \pi_n(o) + \frac{1}{2}p(n-1) = \frac{3}{2}p(n-1)$  for  $t \geq T_{n-1}$

$$\begin{aligned} \sum_{t=1}^{T_n} R_g(t) &= \sum_{k=1}^n \sum_{s=1}^{g(k)} R_g(T_{k-1} + s) \leq g(1) + \sum_{k=2}^n \sum_{s=1}^{g(k)} R_f(T_{k-1} + s) \leq \\ &g(1) + \sum_{k=2}^n \sum_{s=1}^{g(k)} \frac{3}{2}p(k-1) = g(1) + \frac{3}{2} \sum_{k=2}^n g(k)p(k-1) \end{aligned}$$

Particularly, remark  
 $X_t \sim P_n$  but  $Y_t \sim P_{n-1}$   
for  $t \in [T_{n-1}, T_n)$

Once we obtain Lem. 5, Thm. 4 is clear.

Target. Random walk on  $\{0,1, \dots, N\}^n$  w/ an increasing  $n$

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

w/ a fixed  $N$ .

- $\mathfrak{d}(n) = N^n$ ,
- $G(n)$  is a  $\{0,1, \dots, N\}^n$  skeleton,
- $P(n)$  denotes the lazy simple random walk w/ reflection bound, i.e., move to a neighbor w.p.  $1/4n$ , unless boundary for  $n = 1, 2, \dots$

Lem. 7.

Random walk on  $\{0,1, \dots, N\}^n$  is **weakly LHaGG**.

$\mathfrak{d}\}^3$

Thm. 6. If  $\mathcal{D} = (\mathfrak{d}, G, P)$  satisfies

$$\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{(2N)^n} = \infty$$

then  $o$  is recurrent, otherwise  $o$  is transient.

) steps

## Target. Random walk on $\{0,1, \dots, N\}^n$ w/ an increasing $n$

Let  $\mathcal{D} = (\mathfrak{d}, G, P)$  be a RWoGG where

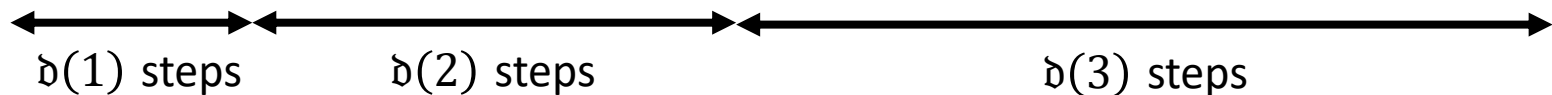
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Random walk on  $\{0,1, \dots, N\}^n$  is **weakly LHaGG**.

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It looks a very simple exercise if you are familiar with **coupling**, but  $n_t \neq m_t$  makes some trouble, in general.

## Target. Random walk on $\{0,1, \dots, N\}^n$ w/ an increasing $n$

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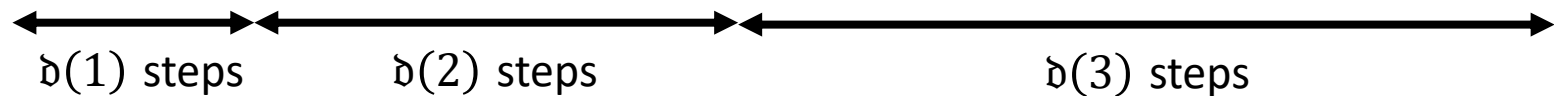
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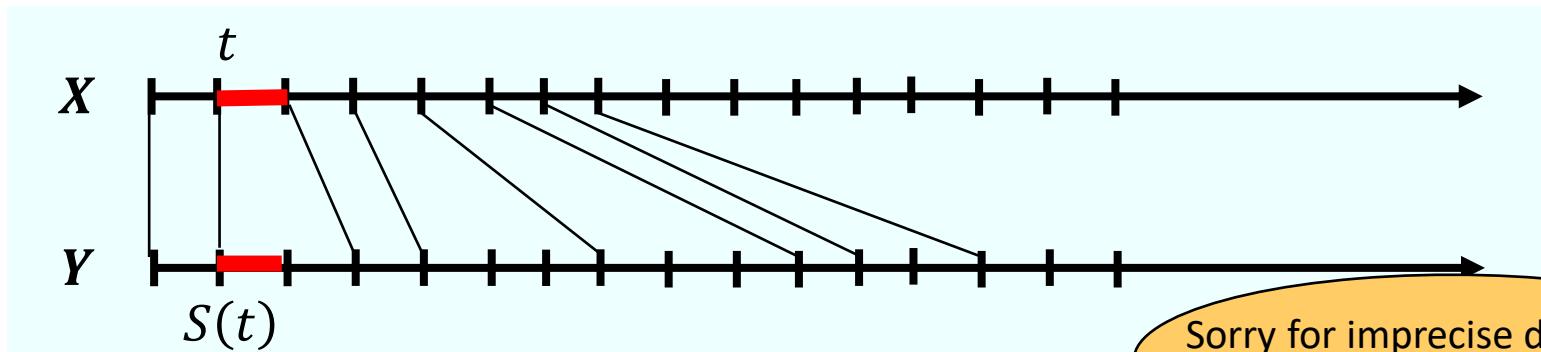


We develop “pausing coupling”

It looks a very simple exercise if you are familiar with **coupling**, but  $n_t \neq m_t$  makes some trouble, in general.

## Figure of pausing coupling

- Let  $\mathbf{X} = X_0, X_1, X_2, \dots \sim \mathcal{D}_f$  and  $\mathbf{Y} = Y_0, Y_1, Y_2, \dots \sim \mathcal{D}_g$   
where let  $\mathcal{D}_g$  grow faster than  $\mathcal{D}_f$ .
- We couple  $\mathbf{X}$  and  $\mathbf{Y}$ , instead of  $X_t$  and  $Y_t$ .



We define time correspondence  $t \mapsto S(t)$  depending on  $\mathbf{Y}$  so that

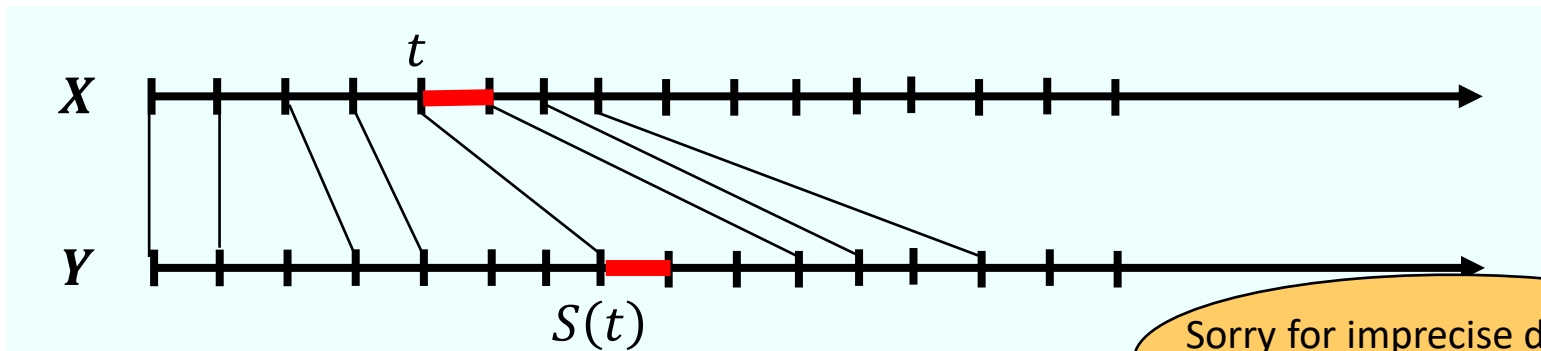
1. if  $Y_t$  does self-loop then so does  $X_{S^{-1}(t)}$ ,
2. if  $Y_t$  updates  $Y_t^i$  for  $i \leq \dim(X_{S^{-1}(t)})$  then  $\mathbf{X}$  updates  $X_{S^{-1}(t)}^i$ ,
3. if  $Y_t$  updates  $Y_t^i$  for  $i > \dim(X_{S^{-1}(t)})$  then  $\mathbf{X}$  pauses.



We need to check “measure conservation” of the coupling.

## Figure of pausing coupling

- Let  $\mathbf{X} = X_0, X_1, X_2, \dots \sim \mathcal{D}_f$  and  $\mathbf{Y} = Y_0, Y_1, Y_2, \dots \sim \mathcal{D}_g$   
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Sorry for imprecise description  
to avoid bothering notation.

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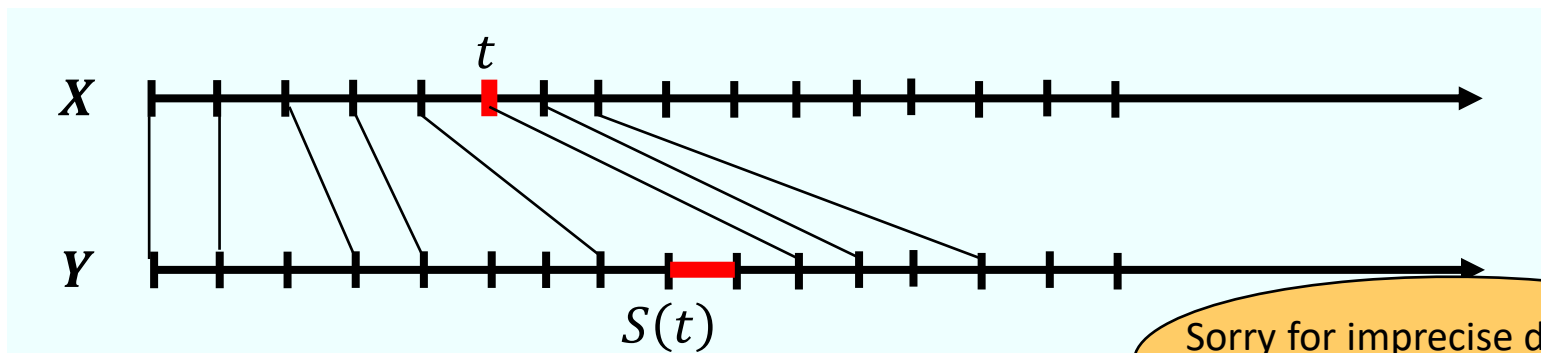
- if  $Y_t$  does self-loop then so does  $X_{S^{-1}(t)}$ ,
- if  $Y_t$  updates  $Y_t^i$  for  $i \leq \dim(X_{S^{-1}(t)})$  then  $X$  updates  $X_{S^{-1}(t)}^i$ ,
- if  $Y_t$  updates  $Y_t^i$  for  $i > \dim(X_{S^{-1}(t)})$  then  $X$  **pauses**.



We need to check “measure conservation” of the coupling.

## Figure of pausing coupling

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We need to check “measure conservation” of the coupling.

## Outline of the proof

Let  $\eta: \mathbf{Y} \mapsto \mathbf{X} = \eta(\mathbf{Y})$  denote the coupling described in the previous slide.

We prove two things:

□ The coupling  $\eta$  preserves the measure, i.e.,

$$\Pr[\mathbf{Y} = \mathbf{y}] = \Pr[\mathbf{X} = \eta(\mathbf{y})]$$

□ The coupling  $\eta$  preserves  $|X_t| \leq |Y_s|$  (meaning “ $|\eta(y_s)| \leq |y_s|$ ”)

for any  $s$  satisfying  $S(t) \leq s < S(t + 1)$ .

➤ This implies  $\#\{t \leq T \mid X_t = o\} \geq \#\{t \leq T \mid Y_t = o\}$  for any  $T$ .



## Def. $S(t)$

Proof.

Suppose  $\mathbf{Y} = Y_0, Y_1, Y_2, Y_3, \dots$  is represented by

$$\boldsymbol{\theta}_Y = (\lambda_1, j_1, \rho_1), (\lambda_2, j_2, \rho_2), (\lambda_3, j_3, \rho_3), \dots$$

We define  $S: \mathbb{Z} \rightarrow \mathbb{Z}$  according to  $\boldsymbol{\theta}$ .

Let  $S(1) = \min\{\min\{t > 0 \mid \lambda_t = 0\}, \min\{t > 0 \mid j_t \in n_0\}\}$ .

Recursively, let

$$S(k) = \min\{\min\{t > S(k-1) \mid \lambda_t = 0\}, \min\{t > S(k-1) \mid j_t \in n_{k-1}\}\}$$

where let  $\min\{\emptyset\} = \infty$ .

If  $S(k) = \infty$  then let  $S(k+1) = \infty$ .

For convenience, let  $S^{-1}(t) = k$  for  $t = S(k) < \infty$  ( $k = 1, 2, \dots$ ).

Then, we define  $\mathbf{X} = X_0, X_1, X_2, \dots$  by

$$\begin{aligned} \boldsymbol{\theta}_X &= \left( (\lambda_{S^{-1}(k)}, j_{S^{-1}(k)}, \rho_{S^{-1}(k)}) \right)_{k=1,2,\dots} \\ &= (\lambda_{S^{-1}(1)}, j_{S^{-1}(1)}, \rho_{S^{-1}(1)}), (\lambda_{S^{-1}(2)}, j_{S^{-1}(2)}, \rho_{S^{-1}(2)}), \dots \end{aligned}$$

as far as  $S(k) < \infty$ .

If  $S(k) = \infty$  then generate  $(\lambda'_k, j'_k, \rho'_k)$  and transit to  $X_{k+1}$  according to it.

## Def. $S(t)$

Proof.

Suppose  $\mathbf{Y} = Y_0, Y_1, Y_2, Y_3, \dots$  is represented by

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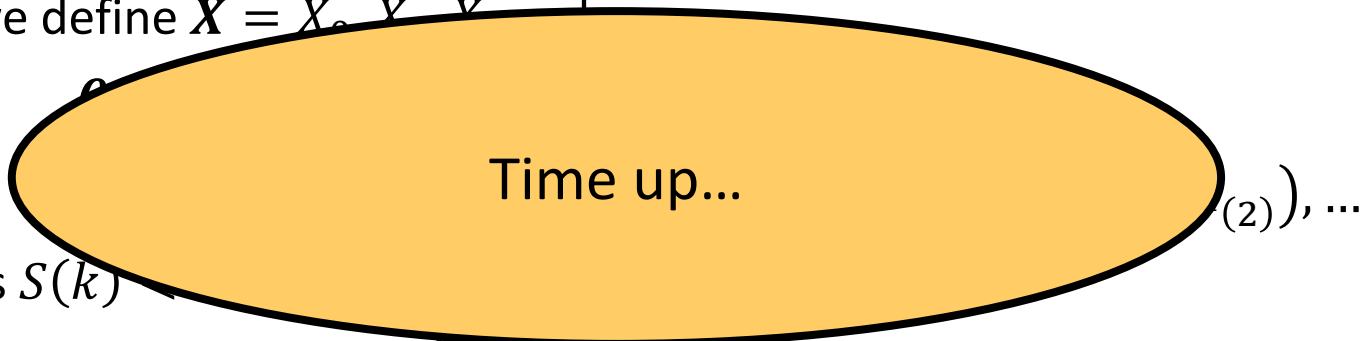
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Then, we define  $\mathbf{X} = X_0, Y_1, Y_2, \dots$



as far as  $S(k)$

If  $S(k) = \infty$  then generate  $(\lambda'_k, j'_k, \rho'_k)$  and transit to  $X_{k+1}$  according to it.



## 5. Concluding remarks

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## Final slide

### Result

- Recurrence/transience of **wLHaGG** RWoGG.
- Random walk on  $\{0,1, \dots, N\}^n$  w/ increasing  $n$  is wLHaGG.
  - Proof by **pausing coupling**.

### Future work

- Simplify the proof
  - Extension to other RWoGGs
    - E.g., GW tree, PA graph, and more general graphs,
    - Edge dynamics, e.g., growing + edge Markovian.
- Analysis of RWoGG beyond recurrence/transience
  - **Hitting time, meeting time, gathering time, etc.**
  - **Find a new limit, undefined for an infinite graph.**



*The end*

---

*Thank you for the attention.*

## Lazy simple random walk on $\{0, 1, \dots, N\}^n$ w/ increasing $n$

Current state  $X_t = (X_t^1, \dots, X_t^{n_t}) \in \{0, 1, \dots, N\}^{n_t}$ .

1. W.p.  $\frac{1}{2}$ , set  $X_{t+1} = X_t$ .
2. Else, choose  $i \in \{1, \dots, n_t\}$  u.a.r.
3. If  $X_t^i$  is not 0 nor  $N$  then update as  $X_{t+1}^i = X_t^i \pm 1$  w.p.  $\frac{1}{2}$  resp.
4. Else if  $X_t^i = 0$  then update as  $X_{t+1}^i = X_t^i + 1$ .
5. Else if  $X_t^i = N$  then update as  $X_{t+1}^i = X_t^i - 1$ .

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5. Else if  $X_t^i = N$  then update as  $X_{t+1}^i = X_t^i - 1$ .

If  $\lambda = 0$  self-loop

Choose  $i$  u.a.r.

If  $\rho = 0$  then  $-1$

A transition  $X_t \mapsto X_{t+1}$  is represented

by uniform r.v.s  $(\lambda, i, \rho) \in \{0,1\} \times \{1, \dots, n_t\} \times \{0,1\}$ .