Approximating Volume ---Randomized vs. Deterministic

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This talk was/is partly supported by MEXT KAKENHI Grant Number 24106005, JSPS KAKENHI Grant Number 25700002, JST PRESTO Grant Number JPMJPR16E4.

Combinatorial Problems and Exercise [Lovasz 1979]

§1. Basic Enumeration

1. In a shop there are k kinds of postcards. We want to send postcards to n friends.

(i) How many different ways can this be done?

-

(ii) What happens if we want to send them different cards?

-

(iii) What happens if we want to send two different cards to each of them (but different persons may get the same card)?

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- $\binom{k}{2}^n$

Combinatorial Problems and Exercise [Lovasz 1979]

§1. Basic Enumeration

2. We have k distinct post cards and want to send them all to our n friends (a friend can get any number of post cards, including 0).

(i) How many ways can this be done?

- ➢

(ii) What happens if we want to send at least one card to each friend?

- ➢

Combinatorial Problems and Exercise [Lovasz 1979]

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- n^k

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- $n! \cdot \left\{ {k \atop n} \right\}$

Stirling number of the second kind $\left\{ {k \atop n} \right\}$ counts the number of ways to partition a set of k elements into n nonempty subsets.

$$\left\{ {k \atop n} \right\} = \left\{ {k-1 \atop n-1} \right\} + n \left\{ {k-1 \atop n} \right\}$$

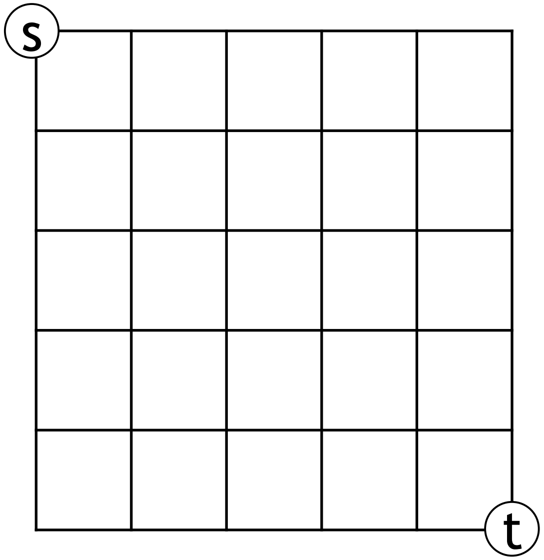
holds. [Wikipedia "Stirling number"]

Another interesting counting problem in [Lovasz 1979]

§1. Basic Enumeration

32. How many shortest paths from s to t in the $n \times n$ grid?

➢



5×5 grid

Another interesting counting problem in [Lovasz 1979]

§1. Basic Enumeration

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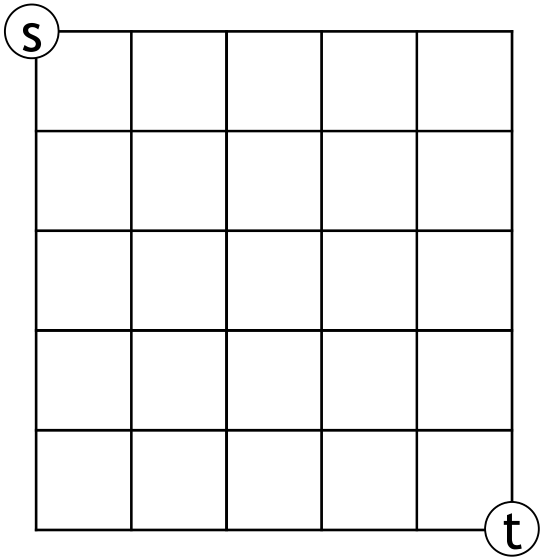
- $\binom{2n}{n}$



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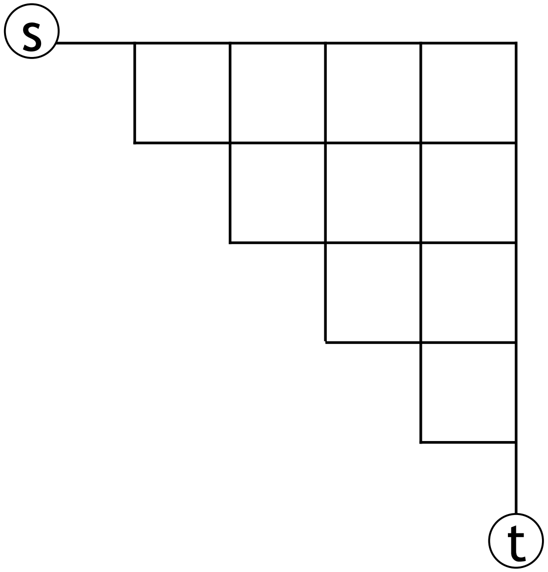
33. How many shortest paths from s to t in the $n \times n$ grid upper than diagonal?

   -

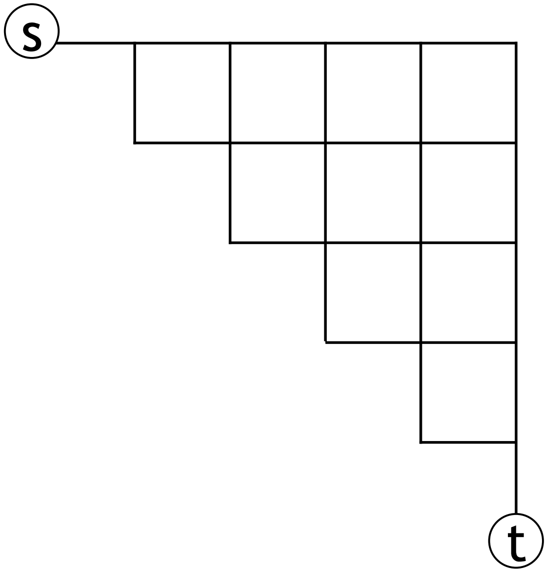


5×5 grid

Another interesting counting problem in [Lovasz 1979]

§1. Basic Enumeration

32. How many shortest paths from s to t in the $n \times n$ grid?
    - $\binom{2n}{n}$
33. How many shortest paths from s to t in the $n \times n$ grid upper than diagonal?
    - $\binom{2n}{n} - \binom{2n}{n-1} = \frac{2n!}{n!(n+1)!}$ (Catalan number)



5 × 5 grid

Counting is a foundation of Combinatorics

#permutations of n elements is $n!$

#k combinations of n elements is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

#Dyck path = #binary trees = #proper parentheses = …

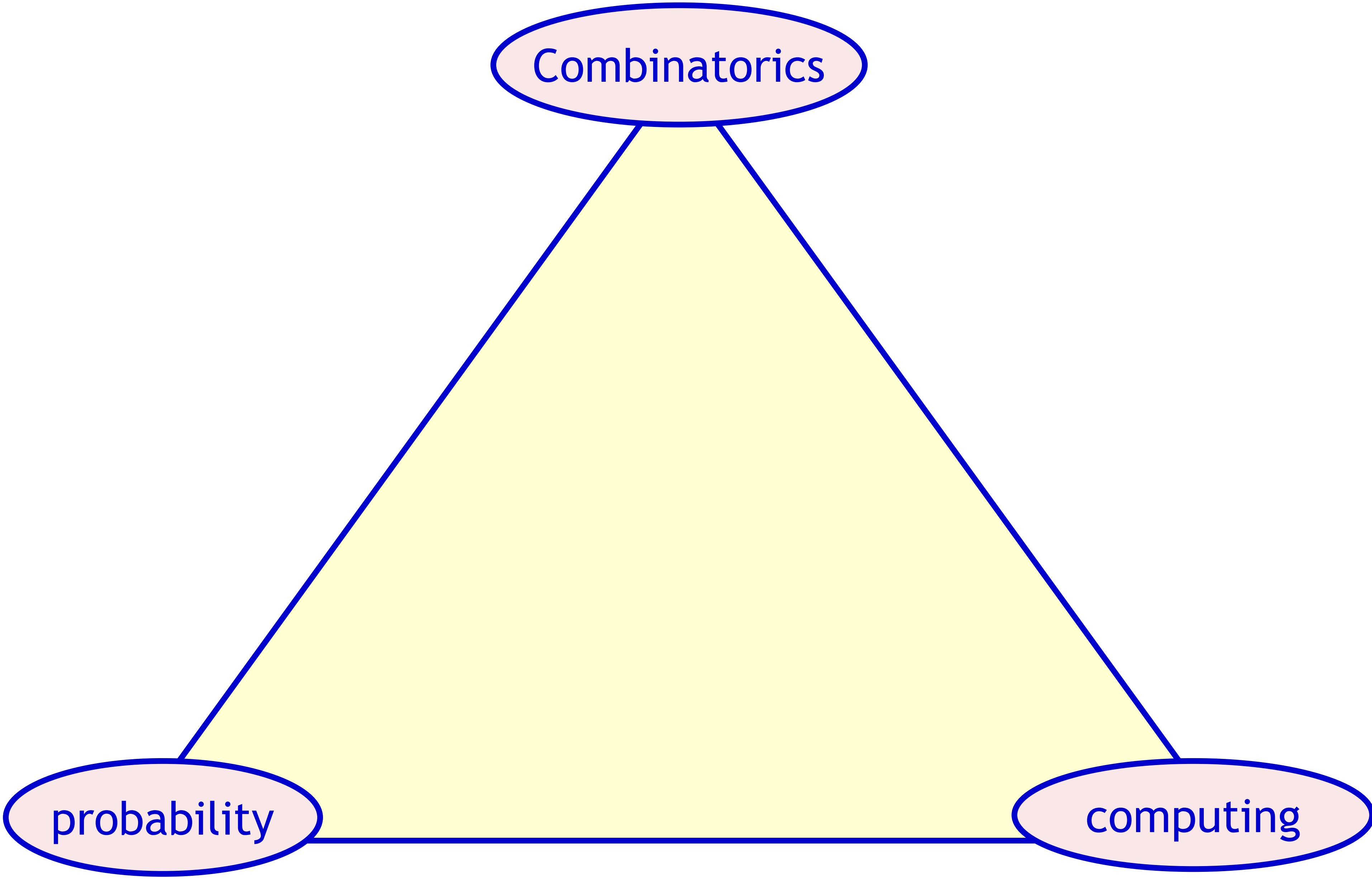
is known as Catalan number = $\binom{2n}{n} - \binom{2n}{n-1} = \frac{2n!}{n!(n+1)!}$

spanning trees $\Rightarrow$ Matrix tree theorem

Many known formula for counting combinatorial objects (while some of them do not have an "explicit form" …such as Stirling number)

… And, many more combinatorial objects for which efficient way to count is not known.

Combinatorics, Probability and Computing are closely related



Combinatorics, Probability and Computing are closely related

Combinatorics

History: mainly from the view point of computing

1979, Valiant, Introduce the class #P

1982, Aldous, coupling method for mixing time

1986, Jerrum, Valiant, Vazirani, relation b/w count & sample

1989, Jerrum, Sinclair, conductance for mixing. (expander)

1989, Toda, $PH \subseteq P^{\#P}$

1991, Dyer, Frieze, Kannan, FPRAS for convex body (MCMC)

1996, Propp, Wilson, perfect sampling (cftp)

1997, Bubley, Dyer, path coupling for mixing time

2004, Jerrum, Sinclair, Vigoda, FPRAS for permanent (MCMC)

probability

computing

Scope of this talk

This talk is concerned with approximate counting/integral

... mainly from the computational view (rather than structure)

Is randomness really necessary for computing?

Talk sketch

0. Introduction
   - Counting is a foundation of Combinatorics
1. Randomized approximation for counting
2. Deterministic approximation of volume I
3. Deterministic approximation of volume II

Is randomness really necessary for computing?

Talk sketch

0. Introduction
1. Randomized approximation for counting

➢ Approximate # simple paths on grid, by MCMC

   i. **Problem description**

   ii. Idea for approximate counting

   iii. How to sample from Ξ_k?

   iv. Then, we counted

2. Deterministic approximation of volume I
3. Deterministic approximation of volume II

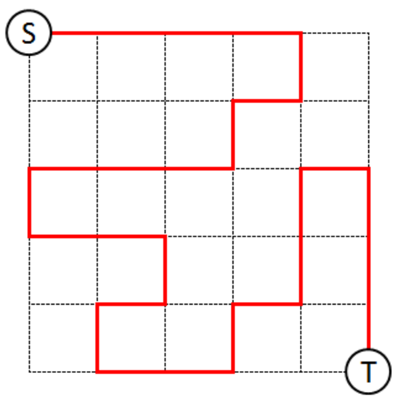
a.k.a. self-avoiding walk

1. Counting simple paths on grid by MCMC

Yuki Shibata, Yukiko Yamauchi,
Shuji Kijima, Masafumi Yamashita
Kyushu Univ.

Counting self-avoiding walk (おねえさん問題 in JP)

Q. How many simple paths on $n \times n$ grid from the NW corner (s) to the SE corner (t).



Minato ERATO, "The Art of 10^64 -Understanding Vastness-" Time with class! Let's count!, YouTube, 2012/9/10,
https://www.youtube.com/watch?v=Q4gTV4r0zRs

MINATO ERATO youtube | search

0	1
1	2
2	12
3	184
4	8512
5	1262816
6	575780564
7	789360053252
8	3266598486981642
9	41044208702632496804
10	1568758030464750013214100
11	182413291514248049241470885236
12	64528039343270018963357185158482118
13	69450664761521361664274701548907358996488
14	227449714676812739631826459327989863387613323440
15	2266745568862672746374567396713098934866324885408319028
16	68745445609149931587631563132489232824587945968099457285419306
17	6344814611237963971310297540795524400449443986866480693646369387855336
18	1782112840842065129893384946652325275167838065704767655931452474605826692782532
19	1523344971704879993080742810319229690899454255323294555776029866737355060592877569255844
20	3962892199823037560207299517133362502106339705739463771515237113377010682364035706704472064940398
21	31374751050137102720420538137382214513103312193698723653061351991346433379389385793965576992246021316463868
22	755970286667345339661519123315222619353103732072409481167391410479517925792743631234987038883317634987271171404439792
23	55435429355237477009914318489061437930690379970964331332556958646484008407334885544566386924020875711242060085408513482933945720
24	12371712231207064758338744862673570832373041989012943539678727080484951695515930485641394550792153037191858028212512280926600304581386791094
25	8402974857881133471007083745436809127296054293775383549824742623937028497898215256929178577083970960121625602506027316549718402106494049978375604247408
26	17369931586279272931175440421236498900372229588288140604663703720910342413276134762789218193498006107082296223143380491348290026721931129627708738890853908108906396

Facts, and our target

- ✓ We don't know any efficient way to calculate the number, say poly(n) time, even for approximation.
- ✓ The number for $n = 26$ is roughly 1.74×10^{163}, where the exact value is presented by [Iwashita et al. 2013], which is the state of the art for exact counting.
- ✓ Counting simple paths in general planer graph is #P-hard [Provan 1986]

□ We in this talk will approximately count it by MCMC.

*H. Iwashita, Y. Nakazawa, J. Kuwahara, T. Uno, S. Minato, Efficient computation of the number of paths in a grid graph with minimal perfect hash functions, Hokkaido University TCS Technical Report, TCS-TR-A-13-64, 2013.

Talk sketch

Two basic idea

What we want is $|\Omega|$ $(= |\Xi_{L^*}|)$.

Notations

- Ω = {s-t simple paths}
- Ξ_k = {s-t simple paths with length at most k}

  $(k = 2n, 2n+2, 2n+4, \dots, L^*)$

  where L^* denotes the length of the longest path:

$$L^* = \begin{cases} (n+1)^2 - 1 & (\text{if } n \text{ is odd}) \\ (n+1)^2 - 2 & (\text{if } n \text{ is even}) \end{cases}$$

Self-reducibility

$$|\Xi_{L^*}| = \frac{|\Xi_{L^*}|}{|\Xi_{L^*-2}|} \cdot \frac{|\Xi_{L^*-2}|}{|\Xi_{L^*-4}|} \cdots \frac{|\Xi_{2n+4}|}{|\Xi_{2n+2}|} \cdot \frac{|\Xi_{2n+2}|}{|\Xi_{2n}|} \cdot |\Xi_{2n}|$$

Idea 2.

inverse $\frac{|\Xi_{k-2}|}{|\Xi_k|}$ is estimated by a Monte Carlo

Idea 1.

$|\Xi_{2n}|$ = #shortest paths

= $\binom{2n}{n}$

Algorithm

Parameters:

  τ (number of transitions of a Markov chain)

  M (number of samples for Monte Carlo)

Input: n (size of grid).

Output: Z (approximation of s-t paths)

```
Set Z := 1;
For (k = 2n + 2; k < L*; k := k + 2){
    Set X ∈ Ξ_k; (X is init. config. of MC)
    Set S := 0; (S is a counter)
    for(i=0; i<M; i++){
        for(j=0; j<τ; j++){
            Update X (Markov chain)
        }
        if(X ∈ Ξ_{k-2}) S++;
    }
    Set Z := Z * M/S ;
}
Output Z;
```

Uniform sampling from Ξ_k

Approximation Ratio

Recall

$\Xi_k = \{$s-t simple paths with length at most $k\}$

Thm.

For any ϵ $(0 < \epsilon < 1)$ and δ $(0 < \delta < 1)$,
let $M = 12n^3(2n^2\epsilon^{-1})^2 \ln(n^2\delta^{-1})$ for the number of uniform samples from Ξ_k $(k = 2n, 2n+2, \dots, L^*)$,
then the approximate solution Z satisfies

$$\Pr[(1-\epsilon)|\Omega| \le Z \le (1+\epsilon)|\Omega|] \ge 1 - \delta.$$

How to sample uniformly from Ξ_k?

Markov chain Monte Carlo (MCMC)

Talk sketch

0. Introduction

1. Randomized approximation of counting

➢ Approximate # simple paths on grid, by MCMC

   i. Problem description

   ii. Idea for approximate counting

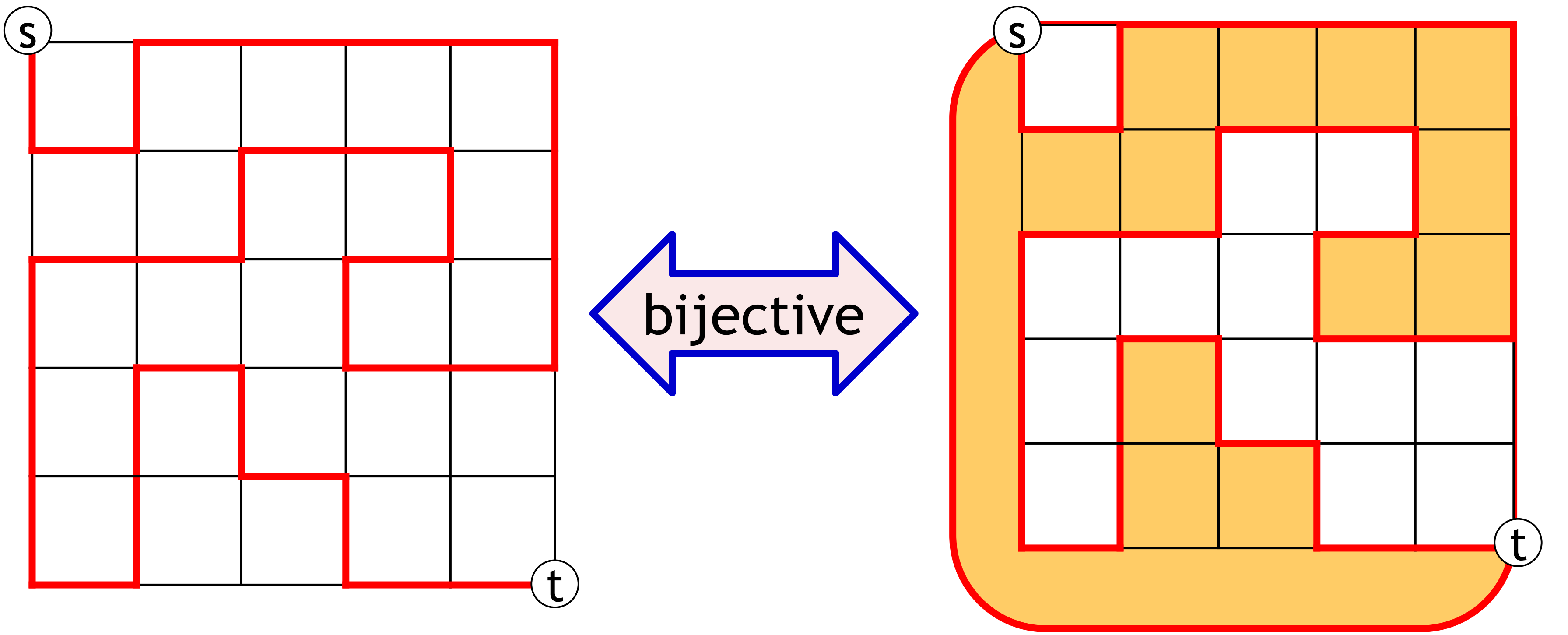
   iii. How to sample from Ξ_k?

   iv. Then, we counted

2. Deterministic approximation of volume I

3. Deterministic approximation of volume II

As a preliminary step, we give a representation of …



s-t simple path

simply connected coloring

Prop.

$$|\Omega|(= |\{s - t \text{ simple paths}\}|)$$
$$= |\{\text{simply connected coloring}\}|$$

Markov chain for simply connected coloring

Markov chain MC

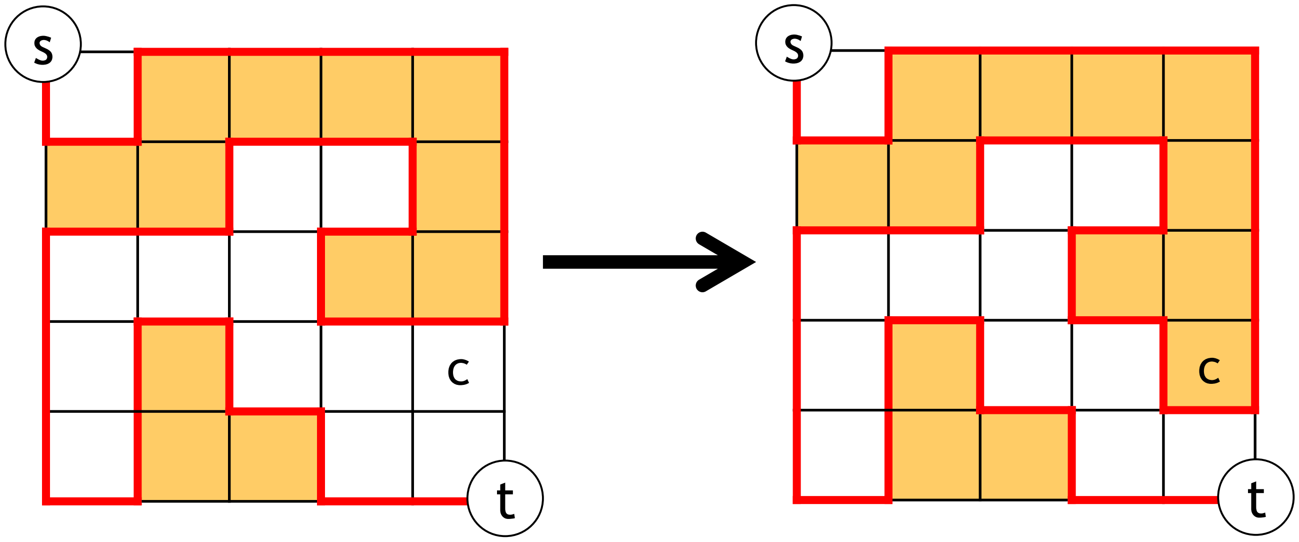
State space: Ξ_k,

Transition $X \to X'$:

Step 1. Choose a cell c u.a.r.

Step 2. Let Y be a state $X \oplus c$.

Step3. If $Y \in \Xi_k$ then set $X' = Y$, else set $X' = X$.



Markov chain for simply connected coloring

<u>Markov chain MC</u>

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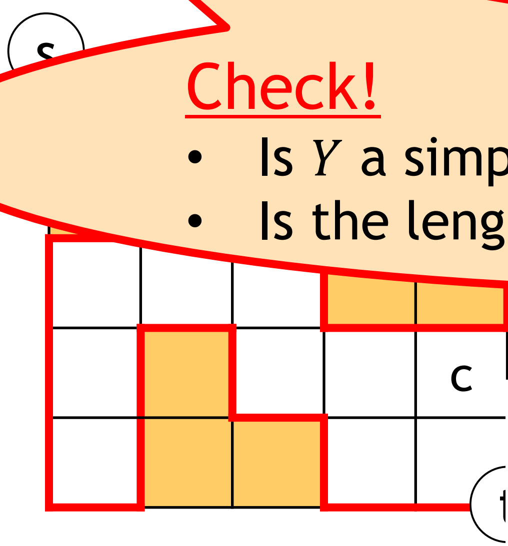
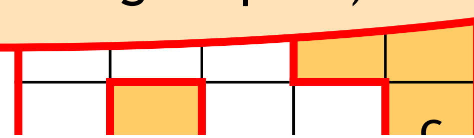
Step3. If $Y \in \Xi_k$ then set $X' = Y$, else set $X' = X$.

<u>Check!</u>

- Is Y a simply connected coloring? and
- Is the length (corresponding s-t path) at most k?





Actual implementation is based on specific case analysis for practical speed up.

Thm.

The MC has the unique limit distribution, which is uniform over Ξ_k

Sketch of proof

- MC is irreducible (transition diagram over Ξ_k is strongly connected)
- MC is aperiodic
- MC satisfies detailed balanced equation

$$\forall X, Y \in \Xi_k, \Pr(X \to Y) = \Pr(Y \to X)$$

Foundations of the MCMC

- irreducible and aperiodic finite Markov chain has the unique stationary distribution.
- detailed balanced equation

$$\forall X, Y \in S, \Pr(X \to Y) = \Pr(Y \to X)$$

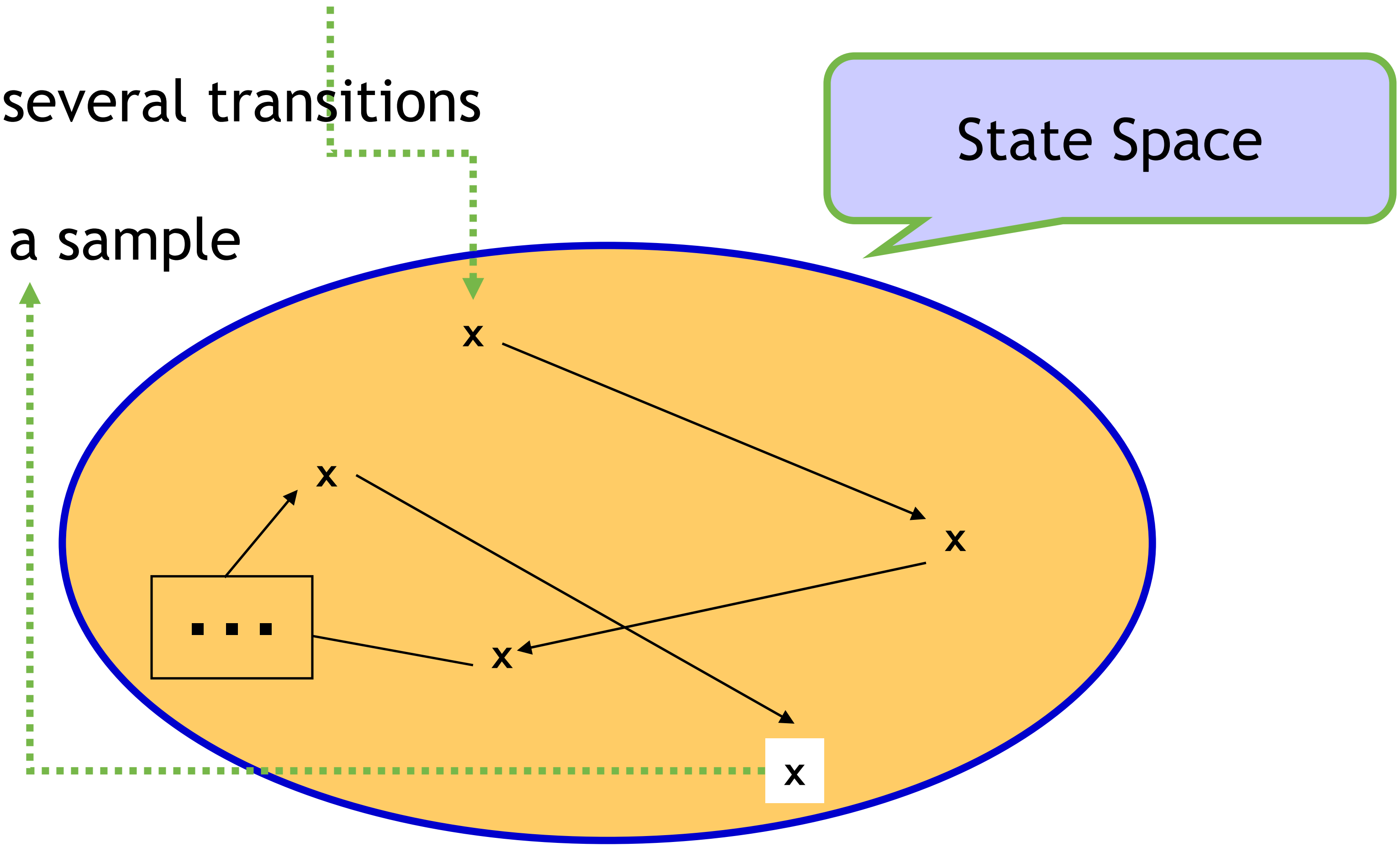
holds, then the stationary distribution is uniform over S.

The idea of “sampling via Markov chain”

Start from arbitrary initial state

Make several transitions

Output a sample



⇒ outputs after many transitions asymptotically according to its stationary distribution

Talk sketch

0. Introduction
1. Randomized approximation of counting

➢ Approximate # simple paths on grid, by MCMC

  i. Problem description
  ii. Idea for approximate counting
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  iv. Then, we approximately counted

2. Deterministic approximation of volume I
3. Deterministic approximation of volume II

Computational results by MCMC

# steps of MC per sample	$\tau = 30$
# samples	$M = 10^7$

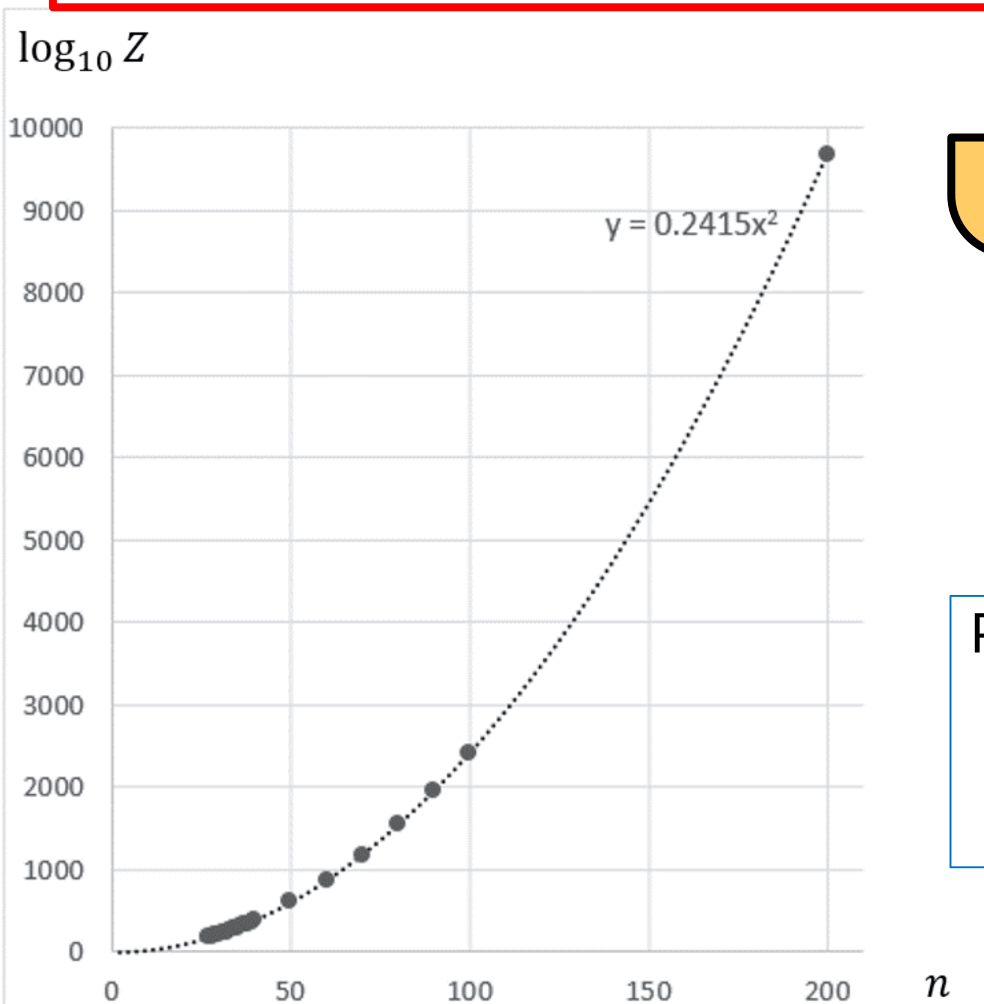
n	True value*	approx.	time
25	$8.40 * 10^{150}$	$8.55 * 10^{150}$	1h 27m
26	$1.74 * 10^{163}$	$1.78 * 10^{163}$	1h 33m
30	unknown	$2.09 * 10^{217}$	2h 4m
50	unknown	$6.35 * 10^{603}$	5h 44m
100	unknown	$6.07 * 10^{2415}$	23h 20m
200	unknown	$1.196 * 10^{9667}$	96h

(approx. is the average of five trials)

We want to tell her …

The number of paths seems about $10^{0.2415n^2} \simeq 1.744^{n^2}$

(conjecture: no proof yet)



Plot of $(n, \log_{10} Z)$

$\sqrt{3}$ conjecture

$$|\Omega| \geq \sqrt{3}^{n^2}$$
$$= (1.732 \dots)^{n^2}$$

Plot of $(n, \log_{10} Z)$

: n = 1-26 true value

: n = 1~200 approx. value

: $y = 0.2415x^2$

Discussion for Section 1: Open Problems

#simple paths (a.k.a. self-avoiding walk)

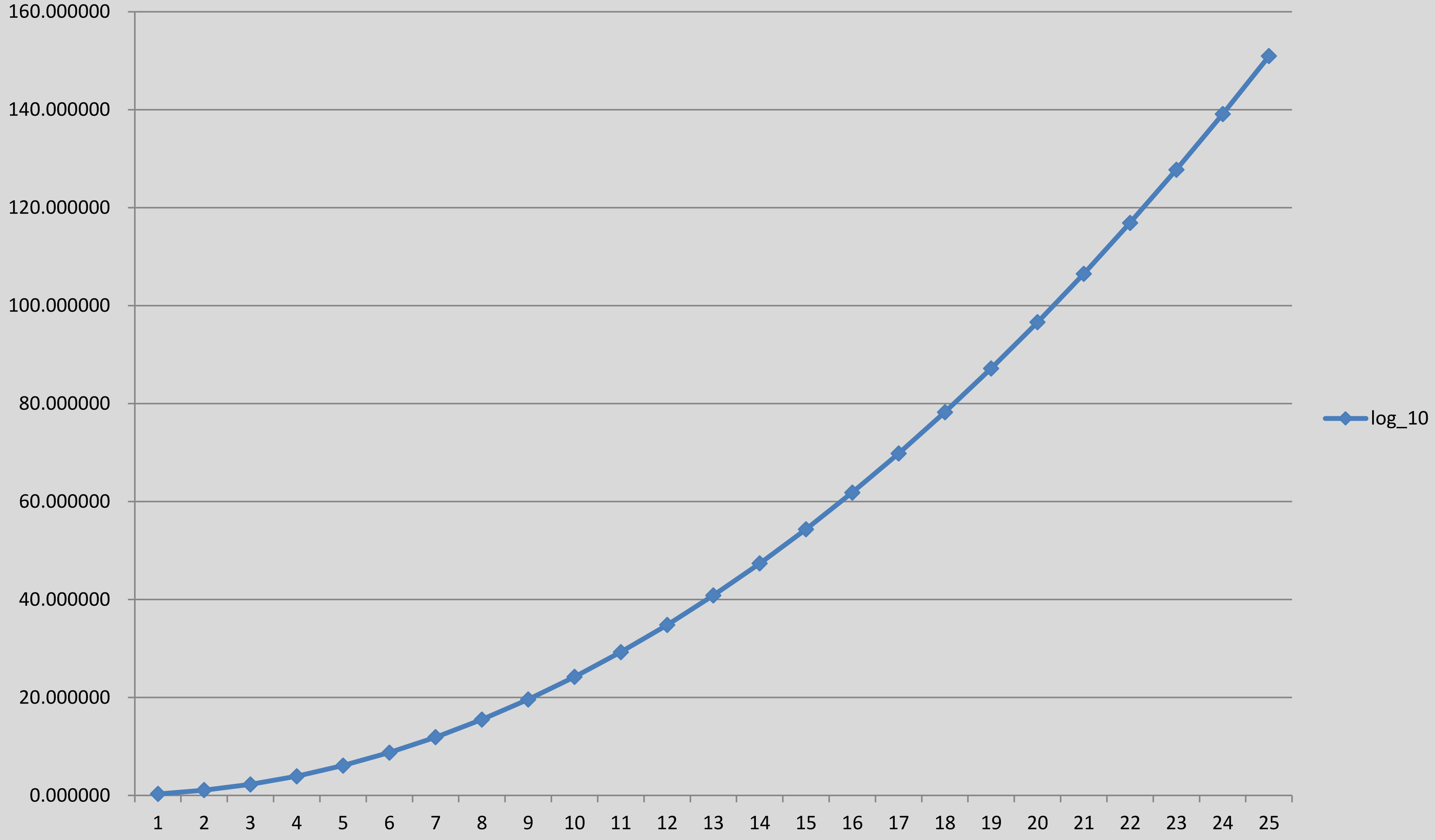
- ☐ Is the mixing time of MC $\mathrm{poly}(n)$?
- ☐ Or, exists (another) $\mathrm{poly}(n)$ time randomized approx. algo.?
- ☐ $\sqrt{3}$-conjecture.
  - ✓ LB 1.628, UB 1.782, [Bousquet-Melos, Guttmann Jensen, 2005]
  - ✓ asymptotically $1.744550 \simeq 10^{0.24168} \leftarrow$ questionable

FPRAS (fully polynomial-time *randomized* approximation scheme)

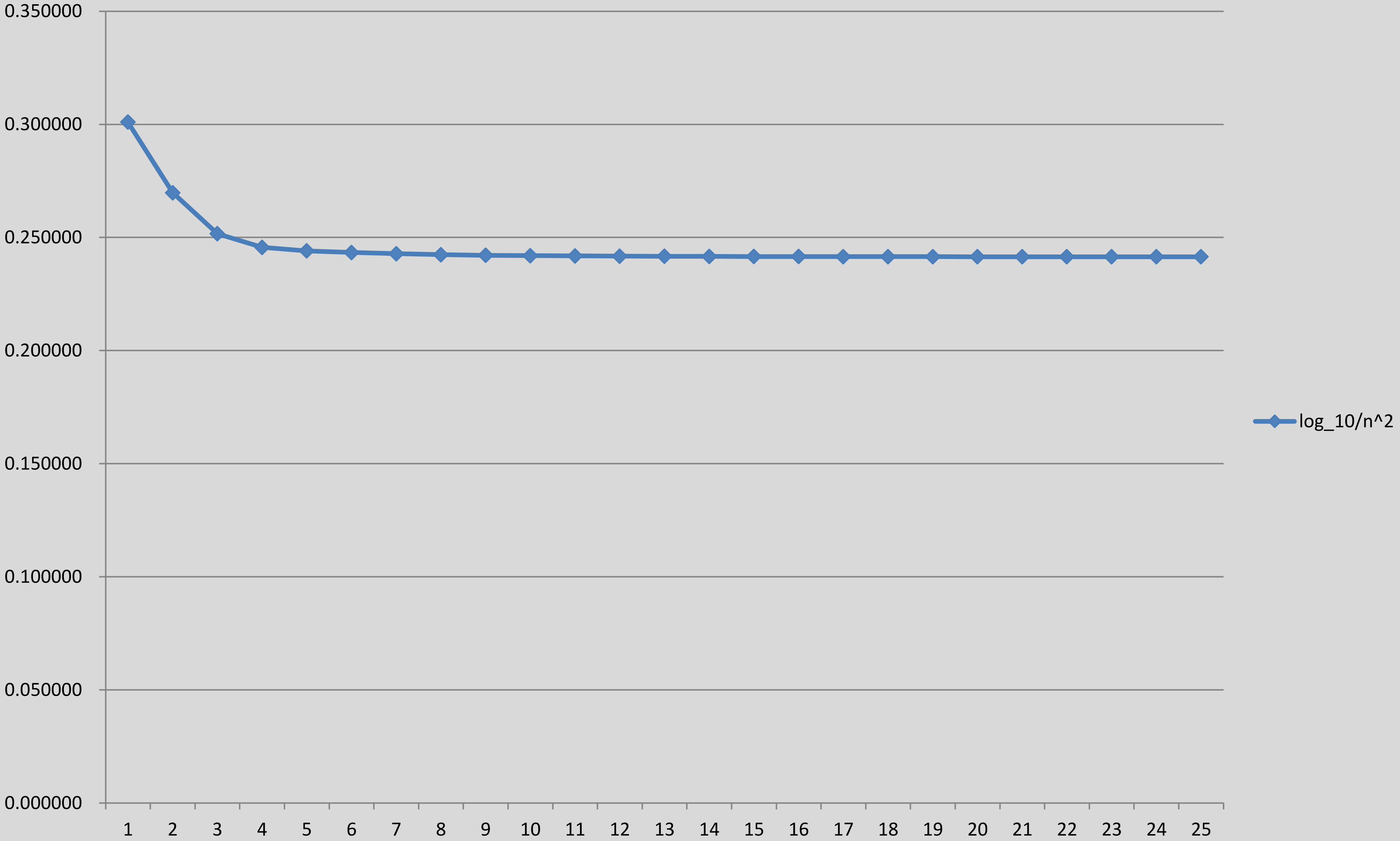
- ☐ #simple paths (in general planer graph)
- ☐ #BIS / #down sets / log-supermodular distribution
- ☐ #forests / Tutte polynomial

	$\|\Omega\|$	$\log_{10}\|\Omega\|$	$\frac{\log_{10}\|\Omega\|}{n^2}$
1	2	0.301030	0.301030
2	12	1.079181	0.269795
3	184	2.264818	0.251646
4	8512	3.930032	0.245627
5	1262816	6.101340	0.244054
6	575780564	8.760257	0.243340
7	7.8936E+11	11.897275	0.242802
8	3.2666E+15	15.514096	0.242408
9	4.10442E+19	19.613252	0.242139
10	1.56876E+24	24.195556	0.241956
11	1.82413E+29	29.261056	0.241827
12	6.4528E+34	34.809748	0.241734
13	6.94507E+40	40.841676	0.241667
14	2.2745E+47	47.356885	0.241617
15	2.26675E+54	54.355403	0.241580
16	6.87454E+61	61.837244	0.241552
17	6.34481E+69	69.802419	0.241531
18	1.78211E+78	78.250935	0.241515
19	1.52334E+87	87.182798	0.241504
20	3.96289E+96	96.598012	0.241495
21	3.1375E+106	106.496580	0.241489
22	7.5597E+116	116.878505	0.241485
23	5.5435E+127	127.743787	0.241482
24	1.2372E+139	139.092430	0.241480
25	8.403E+150	150.924433	0.241479

Plot of $\log_{10}|\Omega|$



Plot of $\frac{\log_{10}|\Omega|}{n^2}$



So what?

MCMC is a powerful and useful technique

for *randomized* approximate counting/integral.

…However, “Is randomness really necessary for computing?”

Talk sketch

0. Introduction
1. Randomized approximation of counting
2. Deterministic approximation of volume I

➢ FPTAS for the volume of 0-1 knapsack polytope

   i. Problem description
   ii. Convolution for the exact volume
   iii. Riemann sum for approximate convolution
   iv. Analysis

3. Deterministic approximation of volume II

2. Deterministic Approximation of the volume of a 0-1 knapsack polytope

Ei Ando (Sojo Univ), **Shuji Kijima** (Kyushu Univ.)

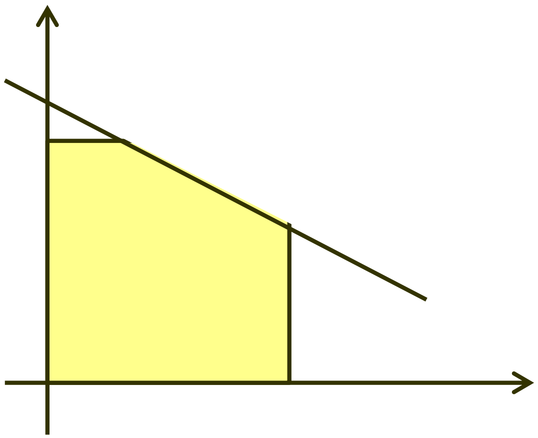
Ei Ando and Shuji Kijima, An FPTAS for the volume computation of 0-1 knapsack polytopes based on approximate convolution, Algorithmica, 76:4 (2016), 1245--1263.

Ei Ando, Shuji Kijima, An FPTAS for the volume of a V-polytope - it is hard to compute the volume of the intersection of two cross-polytopes,arXiv:1607.06173, 2016.

0-1 knapsack polytope

Input: positive integers $a_1, \dots, a_n, b$

Output: the volume of 0-1 knapsack polytope K
$K = \{\boldsymbol{x} \in [0,1]^n \mid a_1 x_1 + \cdots + a_n x_n \leq b\}$



$n = 2$

Approximating the volume is hard.

Note
Independent from P vs.NP

[Elekes 1986] (cf. [Lovász 1986])

As given a convex body by a membership oracle, no polynomial time deterministic algorithm approximates its volume within the ratio 1.999^n.

- If the convex body is a polytope, then there may be a much better way … [Lovász 1986]

Dyer and Frieze [1988]

Computing the volume of a 0-1 knapsack polytope is #P-hard.

(cf. Counting the number of 0-1 knapsack solutions is #P-hard [Valiant 79])

History: Randomized Approximation (FPRAS)

Fully Polynomial-time
Randomized Approximation Scheme

Convex body

Dyer, Frieze and Kannan [1991]

$O^*(n^{23})$ time (The first FPRAS)

⋮

Lovász and Vempala [2006]

$O^*(n^4)$ time

Markov chain Monte Carlo
(MCMC) method

Cousins and Vempala [2015]

$O^*(n^3)$ time

#0-1 knapsack solutions

Morris and Sinclair [2004]

$\mathrm{poly}(n)$ time (MCMC)

Dyer [2003]

$O^*(n^{2.5})$ time (**dynamic programming**)

History Deterministic approximations for #P-hard problems

#0-1knapsack solutions

Dyer [2003]

$\sqrt{n}$ approximation

Gopalan, Klivans and Meka [FOCS 2011]

FPTAS (Fully Polynomial Time Approximation Scheme)

Štefankovič, Vempala and Vigoda [FOCS 2011]

FPTAS based on dynamic programming

Volume of 0-1 knapsack polytope

Li and Shi [2014]

FPTAS $O\left(\frac{n^3}{\epsilon^2}\log\frac{1}{\Delta^2}\log b\right)$ time, based on dynamic programming

Ando and Kijima [2016]

FPTAS $O\left(\frac{n^3}{\epsilon}\right)$ time, based on approximate convolution

Comparison with Li-Shi

Li and Shi [2014]

- ✓ Counting the number of grids in the knapsack polytope (based on the DP by Štefankovič et al.)
- ✓ $O\left(\frac{n^3}{\epsilon^2}\log\frac{1}{\Delta^2}\log b\right)$ time

Ando and Kijima [2016]

- ✓ Approximate convolution (different approach)
- ✓ $O\left(\frac{n^3}{\epsilon}\right)$ time

Talk sketch

0. Introduction

1. Randomized approximation of counting

2. Deterministic approximation of volume I

➢ FPTAS for the volume of 0-1 knapsack polytope

   i. Problem description

   ii. Convolution for the exact volume

   iii. Riemann sum for approximate convolution

   iv. Analysis

3. Deterministic approximation of volume II

0-1 knapsack polytope

Input: positive integers $a_1, \dots, a_n, b$

Output: the volume of 0-1 knapsack polytope K

$$K = \{\boldsymbol{x} \in [0,1]^n \mid a_1 x_1 + \cdots + a_n x_n \leq b\}$$

Compute Vol(K) is #P-hard

Dyer and Frieze [1988]

Thm. [Ando & Kijima 16]

For any ϵ $(0 < \epsilon < 1)$, there exists an algorithm which outputs Z satisfying

$$(1-\epsilon)\mathrm{Vol}(K) \leq Z \leq (1+\epsilon)\mathrm{Vol}(K)$$

in $\mathrm{O}\left(\frac{n^3}{\epsilon}\right)$ time.

As a preliminary step, Normalize knapsack coefficients

For convenience, we normalize coefficients:

Let $\tilde{a}_j = \frac{a_j}{b} M$, and let

$$\tilde{K} \coloneqq \{\, \boldsymbol{x} \in [0,1]^n \mid \tilde{\boldsymbol{a}}^\top \boldsymbol{x} \le M \,\}.$$

Recall

$$K \coloneqq \{\, \boldsymbol{x} \in [0,1]^n \mid \boldsymbol{a}^\top \boldsymbol{x} \le b \,\}.$$

<u>Prop.</u>

$$K = \tilde{K}$$

$M \in \mathbb{Z}_{>0}$ is a parameter for approximation

Convolution for $\mathrm{Vol}(\widetilde{K})$

Def.

Let

$$\Phi_0(y) = \begin{cases} 0, & y < 0 \\ 1, & y \geq 0 \end{cases}$$

and recursively (w.r.t. j) let

$$\Phi_j(y) := \int_0^1 \Phi_{j-1}\left(y - \widetilde{a}_j s\right) \mathrm{d}s$$

convolution at j-th dim.

Prop.

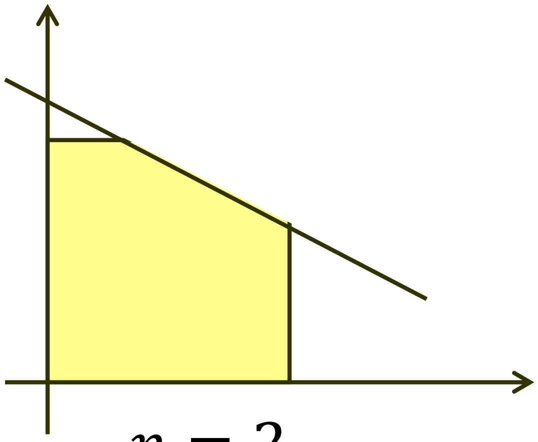
$$\Phi_n(M) = \mathrm{Vol}\left(\widetilde{K}\right)$$

Figure for the inductive convolution

$$\Phi_j(y) := \int_0^1 \Phi_{j-1}(y - \tilde{a}_j s)\mathrm{d}s$$

$$\widetilde{K}_j[s] := \{ (x_1, \dots, x_{j-1}, x_j) \in \widetilde{K}_j \mid x_j = s \}$$

$$\mathrm{Vol}_j \left(\widetilde{K}_j\right) = \int_0^1 \mathrm{Vol}_{j-1}\left(\widetilde{K}_j[s]\right)\mathrm{d}s$$



$n = 2$

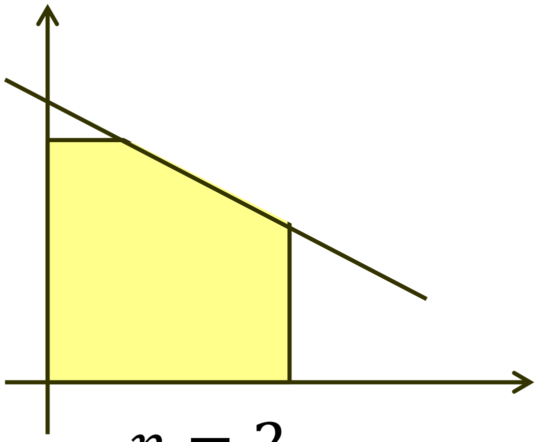
Figure for the inductive convolution

$$\Phi_j(y) := \int_0^1 \Phi_{j-1}(y - \tilde{a}_j s)\mathrm{d}s$$

$$\begin{aligned}\tilde{K}_j[s] &:= \{ (x_1, \dots, x_{j-1}, x_j) \in \tilde{K}_j \mid x_j = s \} \\ &= \{ (x_1, \dots, x_{j-1}, x_j) \in [0,1]^j \mid \tilde{a}_1 x_1 + \cdots + \tilde{a}_{j-1} x_{j-1} + \tilde{a}_j x_j \le y, x_j = s \} \\ &= \{ (x_1, \dots, x_{j-1}, s) \in [0,1]^j \mid \tilde{a}_1 x_1 + \cdots + \tilde{a}_{j-1} x_{j-1} \le y - \tilde{a}_j s \}\end{aligned}$$

$$\mathrm{Vol}_j\left(\tilde{K}_j\right) = \int_0^1 \mathrm{Vol}_{j-1}(\tilde{K}_j[s])\mathrm{d}s$$

$$\Phi_{j-1}(y - \tilde{a}_j s) = \mathrm{Vol}_{j-1}(\tilde{K}_j[s])$$



$n = 2$

Proof Sketch

Prop.

$$\Phi_j(y) \coloneqq \Pr\left[\tilde{a}_1 X_1 + \cdots + \tilde{a}_j X_j \le y\right]$$

Proof Sketch (recursion)

Let

$$\Phi_0(y) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

(indicator function),

$$\Phi_1(y) = \int_0^1 \Phi_0(y - \tilde{a}_1 s)ds = \Pr[y - \tilde{a}_1 X_1 \ge 0] = \Pr[\tilde{a}_1 X_1 \le y]$$

and we obtain the claim for $j = 1$.

Proof Sketch

Prop.

$$\Phi_j(y) \coloneqq \Pr\left[\tilde{a}_1 X_1 + \cdots + \tilde{a}_j X_j \le y\right]$$

f: uniform density on [0,1]

$$f(s) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Recursively assuming the claim when $j-1$

$$\begin{aligned}
&\Pr\left[\tilde{a}_1 X_1 + \cdots + \tilde{a}_{j-1} X_{j-1} + \tilde{a}_j X_j \le y\right] \\
&= \int_{-\infty}^{\infty} \Pr\left[\,\tilde{a}_1 X_1 + \cdots + \tilde{a}_{j-1} X_{j-1} + \tilde{a}_j X_j \le y \mid X_j = s\,\right] f(s)\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \Pr\left[\tilde{a}_1 X_1 + \cdots + \tilde{a}_{j-1} X_{j-1} + \tilde{a}_j s \le y\right] f(s)\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \Pr\left[\tilde{a}_1 X_1 + \cdots + \tilde{a}_{j-1} X_{j-1} \le y - \tilde{a}_j s\right] f(s)\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \Phi_{j-1}\left(y - \tilde{a}_j s\right) f(s)\mathrm{d}s \\
&= \int_{0}^{1} \Phi_{j-1}\left(y - \tilde{a}_j s\right)\mathrm{d}s \\
&= \Phi_j(y)
\end{aligned}$$

Talk sketch

Approximation of Φ by quadrature by parts with G

Definition

Let

$$G_0(y) \coloneqq \Phi_0(y) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Recursively, let

$$\overline{G}_j(y) \coloneqq \int_0^1 G_{j-1}(y - \tilde{a}_j s)\mathrm{d}s$$

and let

$$G_j(y) \coloneqq \overline{G}_j(\lceil y \rceil)$$

Recall

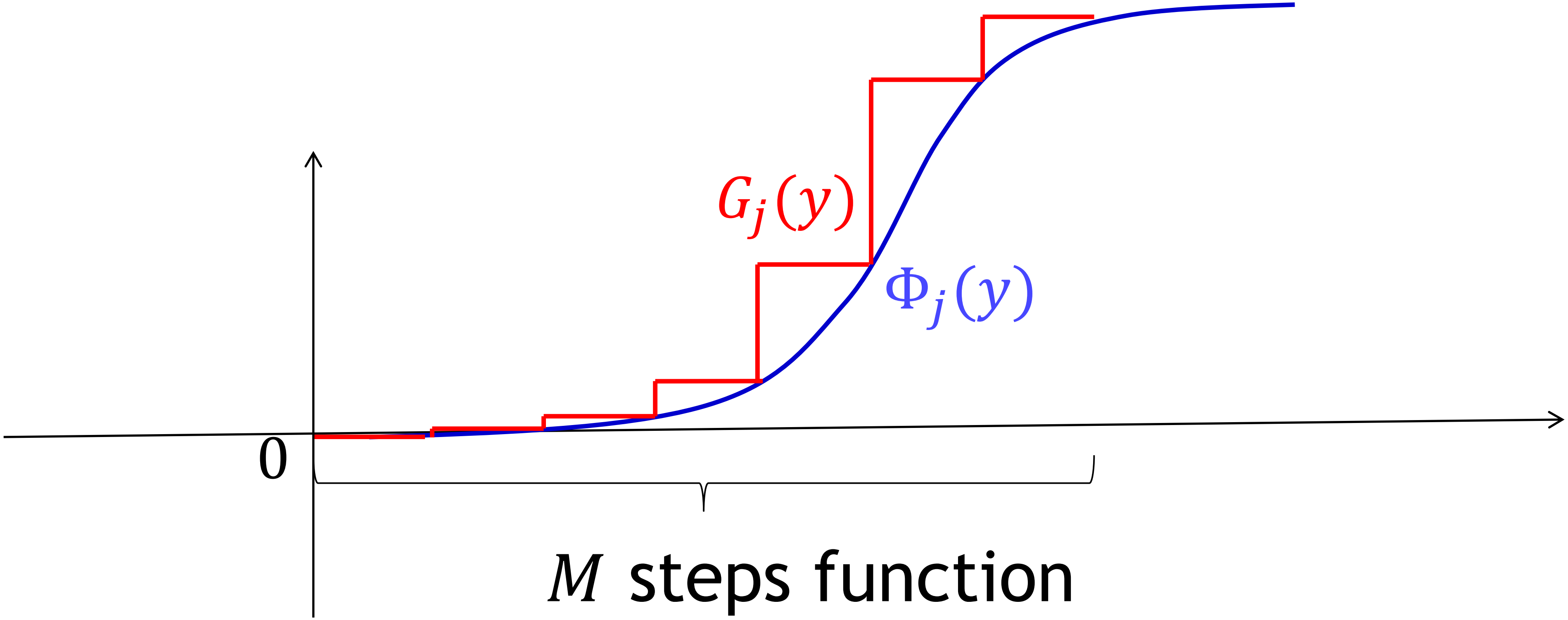
Function $\Phi_j \quad (j = 0,1,\ldots,n)$

$$\Phi_0(y) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

and recursively let

$$\Phi_j(y) \coloneqq \int_0^1 \Phi_{j-1}(y - \tilde{a}_j s)\mathrm{d}s$$

Approximation of Φ by quadrature by parts with G



Calculation of approximate function G_j

Recall
$G_j(y) := \overline{G}_j(\lceil y \rceil)$

For $z \in \mathbb{Z}_{>0}$,

$$
\begin{aligned}
G_j(z) &= \int_0^1 G_{j-1}(z - s\tilde{a}_j)\mathrm{d}s \\
&= \frac{1}{\tilde{a}_j} G_{j-1}(z) + \frac{1}{\tilde{a}_j} G_{j-1}(z-1) + \frac{1}{\tilde{a}_j} G_{j-1}(z-2) + \cdots \\
&= \begin{cases} \displaystyle\sum_{l=0}^{|T|-1} \frac{1}{\tilde{a}_j} G_{j-1}(z-l) + \frac{\tilde{a}_j - \lfloor \tilde{a}_j \rfloor}{\tilde{a}_j} G_{j-1}(z - \lfloor \tilde{a}_j \rfloor) & (\text{if } z - \tilde{a}_j > 0) \\ \displaystyle\sum_{l=0}^{|T|-1} \frac{1}{\tilde{a}_j} G_{j-1}(z-l) & (\text{otherwise}) \end{cases}
\end{aligned}
$$

In principle,
$G_j(z)$ for each $z = 0,1,\ldots,M$ is computed from $G_{j-1}(z')$ $(z' = 0,1,\ldots,M)$ in $O(M)$ time (without using $\int$).

Algorithm

Algorithm

INPUT: $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_n) \in \mathbb{Q}^n_{>0}$.

1. Let $G_0(y) := 0$ for $y \leq 0$ and let $G_0(y) := 1$ for $y > 0$.
2. For $j = 1, \dots, n$
3. For $z = 1, \dots, M$
4. Compute $G_j(z)$;
5. Output $G_n(M)$.

Lemma A

$O(nM^2)$ time.

→ sophisticate →

Lemma A'

$O(nM)$ time.

Lemma B

For any ϵ $(0 < \epsilon \leq 1)$, let $M \geq 2n^2\epsilon^{-1}$, then

$$\Phi_n(M) \leq G_n(M) \leq (1+\epsilon)\Phi_n(M).$$

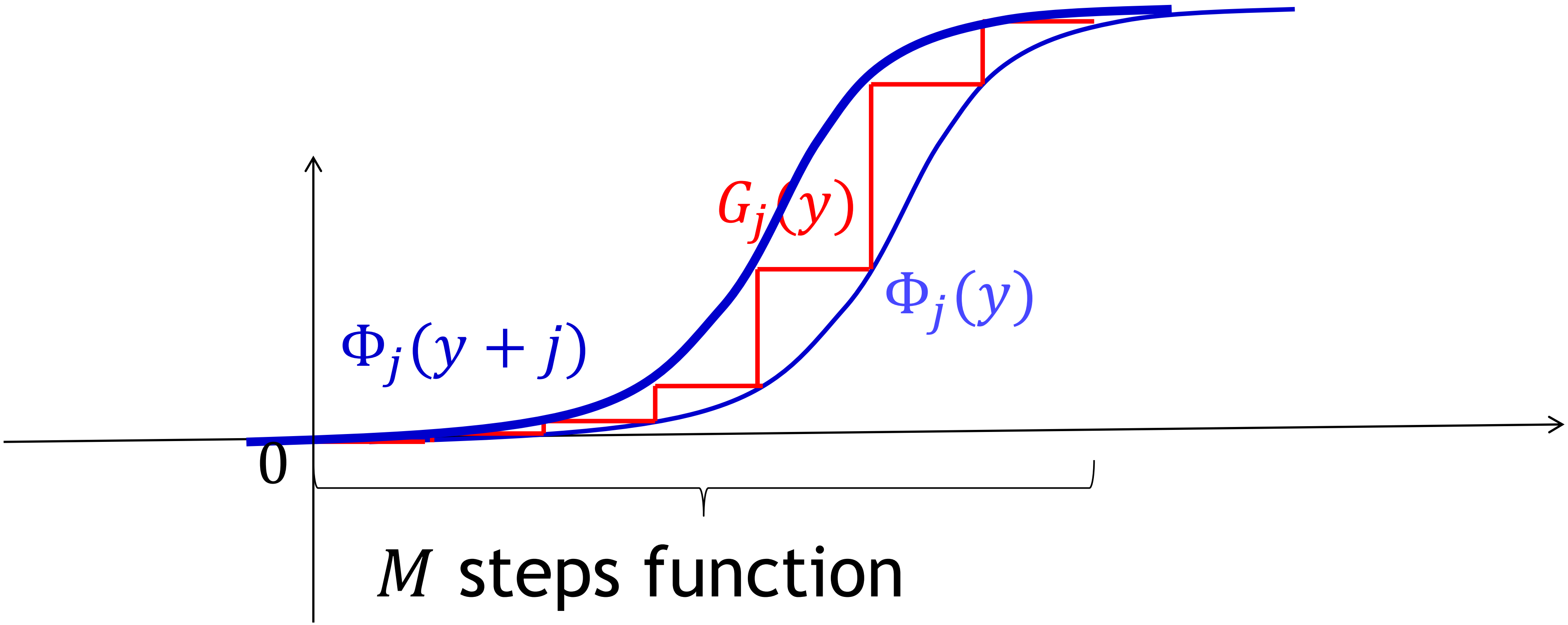
Talk sketch

Tec. 1

Lemma B

For any ϵ $(0 < \epsilon \le 1)$, let $M \ge 2n^2\epsilon^{-1}$, then
$$\Phi_n(M) \le G_n(M) \le (1+\epsilon)\Phi_n(M).$$

Lemma 1 (Horizontal approximation)
$$\Phi_j(y) \le G_j(y) \le \Phi_j(y+j)$$



Approximation ratio

Lemma 1 (Horizontal approximation)

$$\Phi_j(y) \le G_j(y) \le \Phi_j(y+j)$$

Obs. 1

$\Phi_j(y)$, $\overline{G}_j(y)$, $G_j(y)$ are resp. monotone nondecreasing w.r.t. y

Obs. 2

$$\overline{G}_j(y) \le G_j(y) \le \overline{G}_j(y+1)$$

Obs. 3

$$\Phi_j(y) \le \overline{G}_j(y)$$

$$\begin{aligned}\Phi_j(y) &= \int_0^1 \Phi_{j-1}(y - \tilde{a}_j s)\mathrm{d}s \\ &\le \int_0^1 \overline{G}_{j-1}(y - \tilde{a}_j s)\mathrm{d}s \\ &\le \int_0^1 G_{j-1}(y - \tilde{a}_j s)\mathrm{d}s = \overline{G}_j(y)\end{aligned}$$

induct. hypo.

Obs. 2

Obs. 2

$$\overline{G}_j(y) \leq G_j(y) \leq \overline{G}_j(y+1)$$

Tec. 1: Horizontal Approximation

Lemma 1

$$\Phi_j(y) \leq G_j(y) \leq \Phi_j(y+j)$$

Obs. 3

$$\Phi_j(y) \leq \overline{G}_j(y)$$

The former ineq. comes from Obs. 2,3.

Tec. 1: Horizontal Approximation

Obs. 2
$\overline{G}_j(y) \le G_j(y) \le \overline{G}_j(y+1)$

Lemma 1
$$\Phi_j(y) \le G_j(y) \le \Phi_j(y+j)$$

Def.
$$\overline{G}_1(y) := \int_0^1 \Phi_0(y - \tilde{a}_j s)\mathrm{d}s$$

Proof (of the second ineq.)

For $j = 0$, Obs. 2 and $\overline{G}_0(y) = \Phi_0(y)$ implies the claim.

Recursively

$$\overline{G}_j(y') = \int_0^1 G_{j-1}(y' - \tilde{a}_j s)\,ds$$
$$\le \int_0^1 \Phi_{j-1}(y' - \tilde{a}_j s + j - 1)\,ds$$
$$= \Phi_j(y' + (j-1))$$

induct. hypo.

From Obs. 2,

$$G_j(y) \le \overline{G}_j(y+1) \le \Phi_j(y+j)$$

$y' := y + 1$

Analysis of approx. ratio

Lemma B

For any ϵ $(0 < \epsilon \leq 1)$, let $M \geq 2n^2\epsilon^{-1}$, then

$$\Phi_n(M) \leq G_n(M) \leq (1+\epsilon)\Phi_n(M).$$

Lemma 1 (Horizontal approximation)

$$\Phi_j(y) \leq G_j(y) \leq \Phi_j(y+j)$$

$$\Phi_n(M) \leq G_n(M) \leq (1+\epsilon)\Phi_n(M)$$

Analysis of approx. ratio

Lemma B

For any ϵ $(0 < \epsilon \le 1)$, let $M \ge 2n^2\epsilon^{-1}$, then
$$\Phi_n(M) \le G_n(M) \le (1+\epsilon)\Phi_n(M).$$

Lemma 1 (Horizontal approximation)

$$\Phi_j(y) \le G_j(y) \le \Phi_j(y+j)$$

Lemma 2 (cone bound)

$$\frac{\Phi_n(M)}{\Phi_n(M+n)} \ge \left(\frac{M}{M+n}\right)^n$$

$$= \left(\frac{1}{1+\frac{n}{M}}\right)^n \ge \left(1-\frac{n}{M}\right)^n \ge \left(1-\frac{\epsilon}{2n}\right)^n \ge 1 - n\frac{\epsilon}{2n} = 1-\frac{\epsilon}{2} \ge \frac{1}{1+\epsilon}$$

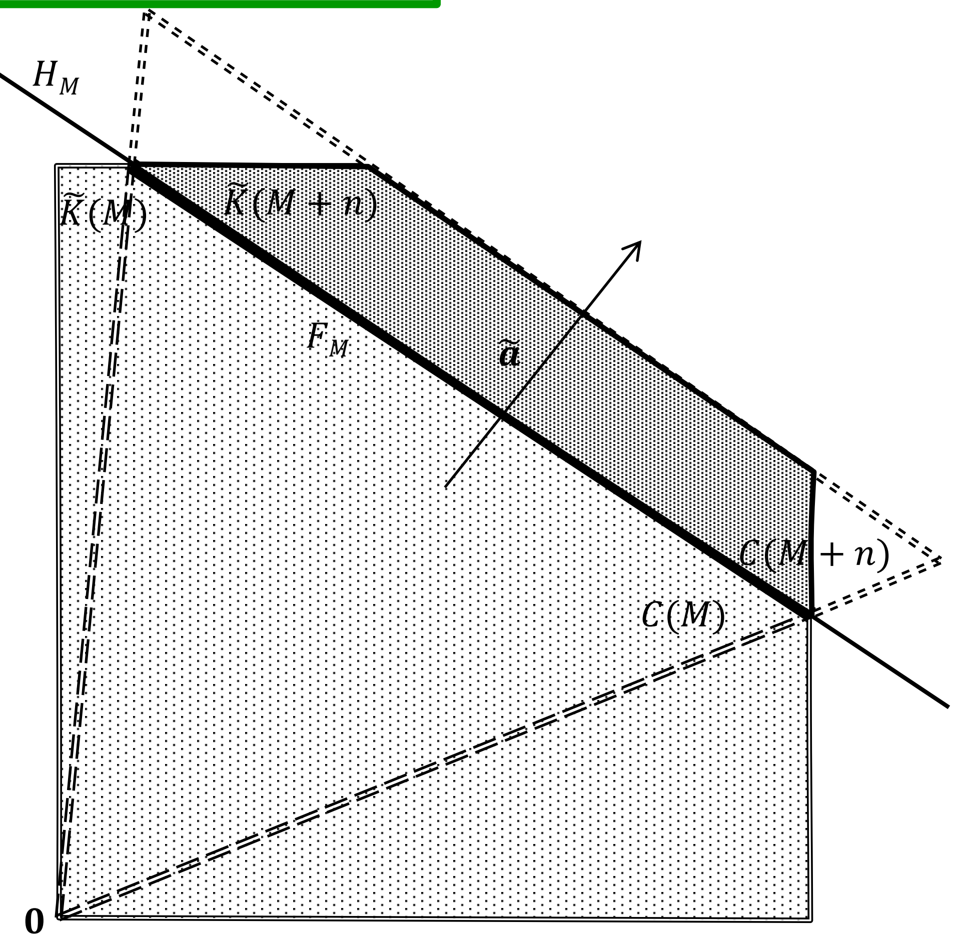
$M \ge 2n^2\epsilon^{-1}$

$$\Phi_n(M) \le G_n(M) \le \Phi_n(M+n) \le (1+\epsilon)\Phi_n(M)$$

Tec. 2: Cone bound (for vertical approx. ratio)

Lemma 2

$$\frac{\Phi_n(M)}{\Phi_n(M+n)} \geq \left(\frac{M}{M+n}\right)^n$$



0-1 knapsack polytope

Input: positive integers $a_1, \dots, a_n, b$

Output: the volume of 0-1 knapsack polytope K

$$K = \{\boldsymbol{x} \in [0,1]^n \mid a_1 x_1 + \cdots + a_n x_n \leq b\}$$

Compute Vol(K) is #P-hard

Dyer and Frieze [1988]

Thm. [Ando & Kijima 16]

For any ϵ $(0 < \epsilon < 1)$, there exists an algorithm which outputs Z satisfying

$$(1-\epsilon)\mathrm{Vol}(K) \leq Z \leq (1+\epsilon)\mathrm{Vol}(K)$$

in $O\left(\frac{n^3}{\epsilon}\right)$ time.

Discussion for Section 2

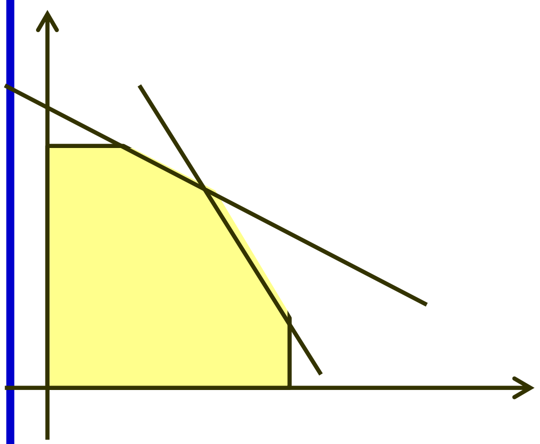
Extension

The algorithm is extend to ones with m constraints (so called "m-D knapsack").

INPUT: m vectors $\boldsymbol{a}_1^\top, \dots, \boldsymbol{a}_m^\top \in \mathbb{Z}_{\geq 0}^n$ and a vector $\boldsymbol{b} \in \mathbb{Z}_{\geq 0}^m$

OUTPUT: $\mathrm{Vol}(K)$ for $K = \{\boldsymbol{x} \in [0,1]^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\}$

where $A = \begin{pmatrix} \boldsymbol{a}_1^\top \\ \vdots \\ \boldsymbol{a}_m^\top \end{pmatrix}$.



$n = 2$

It runs in $O\left(\left(\frac{n^2}{\epsilon}\right)^{m+1} nm \log m\right)$ time

for const. m

$O(n^7\epsilon^{-3})$ when $m = 2$

$O(n^9\epsilon^{-4})$ when $m = 3$

$O(n^{11}\epsilon^{-5})$ when $m = 4$

Future work

Is FPTAS for more general polytope

Talk sketch

0. Introduction
1. Randomized approximation of counting
2. Deterministic approximation of volume I
3. Deterministic approximation of volume II

➢ FPTAS for the Volume of some $\mathcal{V}$-polytope

i. Problem description

ii. Idea: Reduction to $\mathrm{Vol}\big(C(\mathbf{0},1) \cap C(\boldsymbol{c},r)\big)$

iii. Core: FPTAS for $\mathrm{Vol}\big(C(\mathbf{0},1) \cap C(\boldsymbol{c},r)\big)$

iv. #P-hardness of $\mathrm{Vol}\big(C(0,1) \cap C(c,r)\big)$

3. Deterministic Approximation of the volume of some $\mathcal{V}$-polytope

Ei Ando (Sojo Univ), **Shuji Kijima** (Kyushu Univ.)

Ei Ando and Shuji Kijima, An FPTAS for the volume computation of 0-1 knapsack polytopes based on approximate convolution, Algorithmica, 76:4 (2016), 1245--1263.

Ei Ando, Shuji Kijima, An FPTAS for the volume of a V-polytope - it is hard to compute the volume of the intersection of two cross-polytopes,arXiv:1607.06173, 2016.

Talk sketch

0. Introduction
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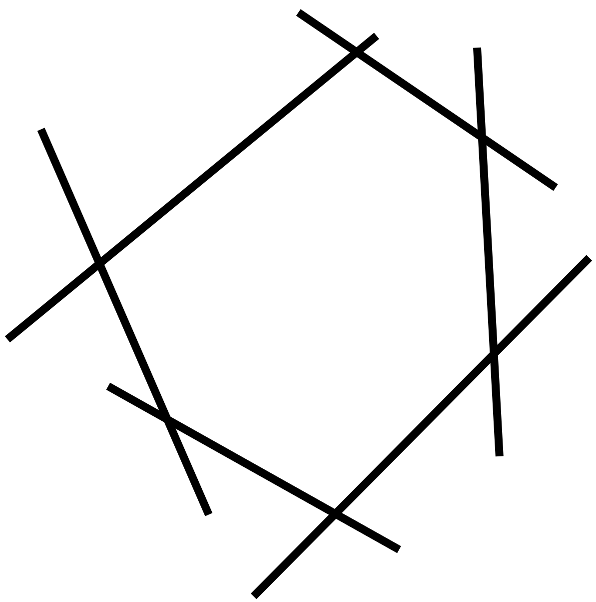
iv. #P-hardness of $\mathrm{Vol}\big(C(0,1)\cap C(c,r)\big)$

H-polytope V-polytope

An $\mathcal{H}$-polytope is an intersection of finitely many closed half-space in $\mathbb{R}^n$.

A $\mathcal{V}$-polytope is a convex hull of a finite point set in $\mathbb{R}^n$.



In 2-D, the difference may seem vague.

Consider n-D hypercube: $2n$ facets and 2^n vertices.

Consider n-D cross-polytope (L_1-ball): 2^nfacets and $2n$ vertices.

Approximating the volume is hard.

Note
Independent from P vs.NP

[Elekes 1986] (cf. [Lovász 1986])

As given a convex body by a membership oracle, no polynomial time deterministic algorithm approximates its volume within the ratio 1.999^n.

- If the convex body is a polytope, then there may be a much better way … [Lovász 1986]

Dyer and Frieze [1988]

Computing the volume of a 0-1 knapsack polytope is #P-hard.

Khachiyan [1989]

Computing the volume of a “polar” knapsack polytope is #P-hard, motivated by the complexity of the volume a $\mathcal{V}$-polytope.

Knapsack "dual" polytope

Input: Positive integers $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$

Output: Volume of the knapsack "dual" polytope $P_{\boldsymbol{a}}$ given by

$$P_{\boldsymbol{a}} \stackrel{\text{def}}{=} \text{conv}\{\pm\boldsymbol{e_1}, \pm\boldsymbol{e_2}, \dots, \pm\boldsymbol{e_n}, \boldsymbol{a}\}$$
$$= \text{conv}\{C(\boldsymbol{0}, 1), \boldsymbol{a}\}$$

where $\boldsymbol{e_i}$ denotes the i-th unit vector.

Notation

For convenience, let

$$C(\boldsymbol{c}, r) \stackrel{\text{def}}{=} \text{conv}\{\, \boldsymbol{c} \pm r\boldsymbol{e_i} \mid i = 1, \dots n \,\}$$
$$= \{\, \boldsymbol{c} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{c}\|_1 \leq r \,\}$$

for $\boldsymbol{c} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{\geq 0}$.

#P-hard [Khachiyan 1989]

Knapsack "dual" polytope

Input: Positive integers $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

Output: Volume of the knapsack "dual" polytope $P_{\boldsymbol{a}}$ given by

$$P_{\boldsymbol{a}} \stackrel{\text{def}}{=} \text{conv}\{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \ldots, \pm \boldsymbol{e}_n, \boldsymbol{a}\}$$
$$= \text{conv}\{C(\boldsymbol{0}, 1), \boldsymbol{a}\}$$

where $\boldsymbol{e}_i$ denotes the i-th unit vector.

Notation

For convenience, let

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$$= \{\, \boldsymbol{c} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{c}\|_1 \leq r \,\}$$

Thm. [Ando & Kijima 16]

For any ϵ $(0 < \epsilon < 1)$, there exists an algorithm which outputs Z satisfying

$$(1 - \epsilon)\text{Vol}(P_{\boldsymbol{a}}) \leq Z \leq (1 + \epsilon)\text{Vol}(P_{\boldsymbol{a}})$$

in $\mathrm{O}\left(\frac{n^{10}}{\epsilon^6}\right)$ time.

#

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iii. FPTAS for $\mathrm{Vol}\big(C(\mathbf{0},1) \cap C(\boldsymbol{c},r)\big)$

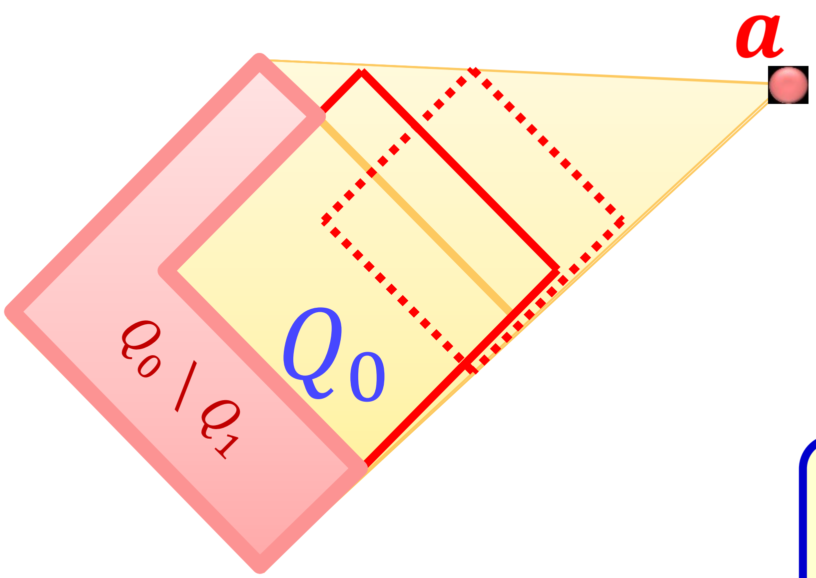
iv. #P-hardness of $\mathrm{Vol}\big(C(0,1) \cap C(c,r)\big)$

Idea

Approximate $P_{\boldsymbol{a}}$ by the union of a geometric series of Crosspolytopes converging to $\boldsymbol{a}$.

$P_{\boldsymbol{a}} = \mathrm{conv}\{C(\mathbf{0},1), \boldsymbol{a}\}$

Let $Q_k \stackrel{\mathrm{def}}{=} C\left((1-\beta^k)\boldsymbol{a}, \beta^k\right)$ for some β $(0<\beta<1)$



i.e.

$Q_0 = C(\mathbf{0},1)$
$Q_1 = C\big((\mathbf{1}-\beta)\boldsymbol{a}, \beta\big)$
$\vdots$
$Q_\infty = C(\boldsymbol{a},0)$

$$\mathrm{Vol}(P_{\boldsymbol{a}}) \simeq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right)$$

Precisely

Union of geometric series of cross-polytopes

Lemma 1

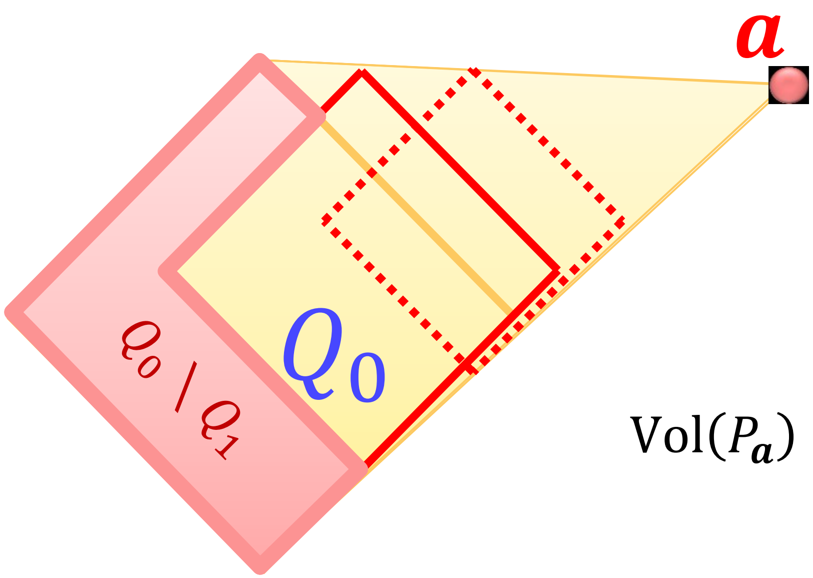
If $1 - \beta \leq \frac{c_1 \epsilon}{n \|\boldsymbol{a}\|_1}$ where $0 < c_1 \epsilon < 1$, then

$$(1 - c_1 \epsilon) \cdot \mathrm{Vol}(P_{\boldsymbol{a}}) \leq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) \leq \mathrm{Vol}(P_{\boldsymbol{a}})$$

Idea

Approximate $P_{\boldsymbol{a}}$ by the union of a geometric series of Crosspolytopes converging to $\boldsymbol{a}$.

Let $Q_k \stackrel{\text{def}}{=} C\left(\left(1-\beta^k\right)\boldsymbol{a}, \beta^k\right)$ for some β $(0 < \beta < 1)$



i.e.

$Q_0 = C(\mathbf{0}, 1)$

$Q_1 = C\big((\mathbf{1}-\beta)\boldsymbol{a}, \beta\big)$

$\vdots$

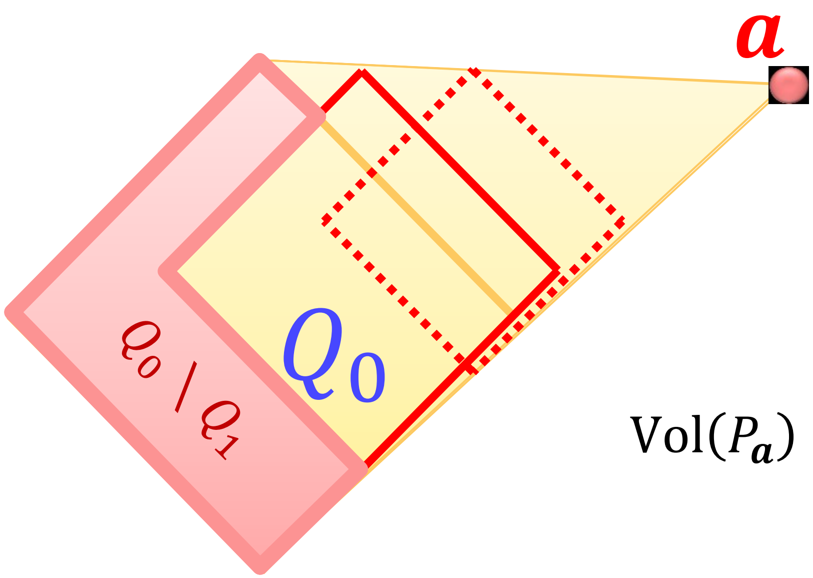
$Q_\infty = C(\boldsymbol{a}, 0)$

$$\mathrm{Vol}(P_{\boldsymbol{a}}) \simeq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right)$$

Idea

Approximate $P_{\boldsymbol{a}}$ by the union of a geometric series of Crosspolytopes converging to $\boldsymbol{a}$.

Let $Q_k \stackrel{\text{def}}{=} C\left(\left(1-\beta^k\right)\boldsymbol{a}, \beta^k\right)$ for some β $(0 < \beta < 1)$



i.e.

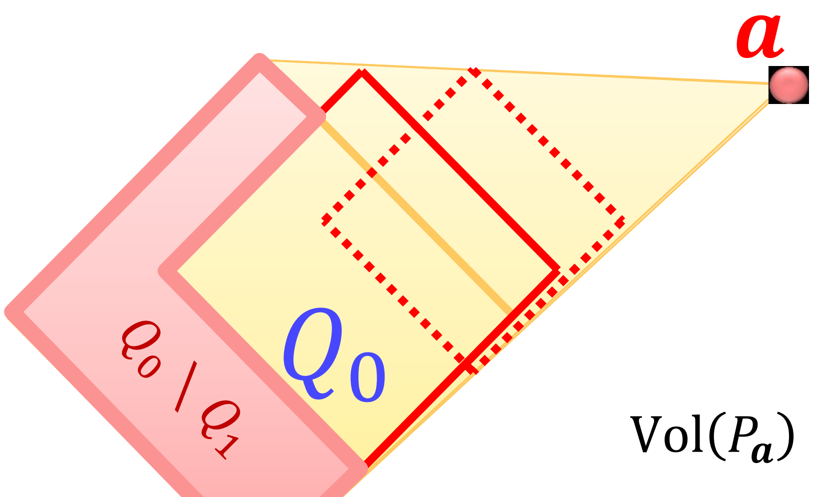
$$Q_0 = C(\mathbf{0}, 1)$$
$$Q_1 = C\left((\mathbf{1}-\beta)\boldsymbol{a}, \beta\right)$$
$$\vdots$$
$$Q_\infty = C(\boldsymbol{a}, 0)$$

$$\begin{aligned}
\mathrm{Vol}(P_{\boldsymbol{a}}) &\simeq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) \\
&= \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} (Q_k \setminus Q_{k+1})\right) \\
&= \sum_{k=0}^{\infty} \mathrm{Vol}(Q_k \setminus Q_{k+1}) \\
&= \sum_{k=0}^{\infty} \beta^k \mathrm{Vol}(Q_0 \setminus Q_1) = \frac{\mathrm{Vol}(Q_0 \setminus Q_1)}{1-\beta^n}
\end{aligned}$$

Idea

Approximate $P_{\boldsymbol{a}}$ by the union of a geometric series of Crosspolytopes converging to $\boldsymbol{a}$.

Let $Q_k \stackrel{\text{def}}{=} C\left(\left(1-\beta^k\right)\boldsymbol{a}, \beta^k\right)$ for some β $(0 < \beta < 1)$



i.e.

$$Q_0 = C(\mathbf{0}, 1)$$
$$Q_1 = C((\mathbf{1}-\beta)\boldsymbol{a}, \beta)$$
$$\vdots$$
$$Q_\infty = C(\boldsymbol{a}, 0)$$

$$\mathrm{Vol}(P_{\boldsymbol{a}}) \simeq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right)$$

$$\bigcup_{k=0}^{\infty} (Q_k \setminus Q_{k+1})\Bigg)$$

$$\mathrm{Vol}(Q_k \setminus Q_{k+1})$$

$$\beta^k \mathrm{Vol}(Q_0 \setminus Q_1) = \frac{\mathrm{Vol}(Q_0 \setminus Q_1)}{1-\beta^n}$$

$Q_0 \setminus Q_1$ is not convex $\Rightarrow$ intractable

$$\mathrm{Vol}(Q_0 \setminus Q_1) = \mathrm{Vol}(Q_0) - \mathrm{Vol}(Q_0 \cap Q_1)$$
$$= \frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)$$

$Q_0 \cap Q_1$ is convex $\Rightarrow$ tractable

Requirements to β are conflict, but is possible to be settled

Lemma 1

If $1-\beta \leq \frac{c_1\epsilon}{n\|\boldsymbol{a}\|_1}$ where $0 < c_1\epsilon < 1$, then

$$(1-c_1\epsilon)\cdot \mathrm{Vol}(P_{\boldsymbol{a}}) \leq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) \leq \mathrm{Vol}(P_{\boldsymbol{a}})$$

Proposition

$$\mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) = \frac{1}{1-\beta^n}\left(\frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)\right)$$

Approximate P_a by the union of a geometric series of

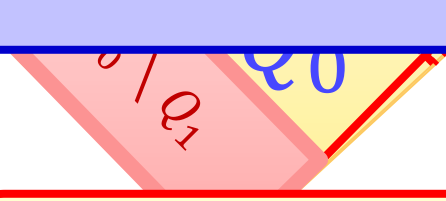
"−" is a dangerous operation in approximation:

For example,

suppose you know $x \simeq 49$ with approximate ratio 1%.

Clearly, $50 + x \simeq 99$ with approximate ratio 1%.

We hope $50 - x \simeq 1$, however it may be 0.5, 0.1, or 0.0001 etc.



$$\mathrm{Vol}(P_a) \simeq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right)$$

$$\bigcup_{k=0}^{\infty} (Q_k \setminus Q_{k+1})\Big)$$

$$\mathrm{Vol}(Q_k \setminus Q_{k+1})$$

$$\beta^k \mathrm{Vol}(Q_0 \setminus Q_1) = \frac{\mathrm{Vol}(Q_0 \setminus Q_1)}{1 - \beta^n}$$

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Approximate $P_{\boldsymbol{a}}$ by the union of a geometric series of Crosspolytopes converging to $\boldsymbol{a}$.

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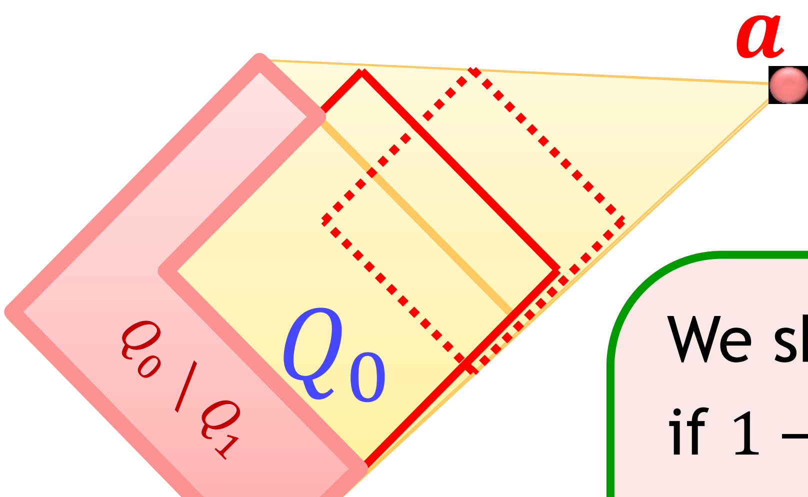

i.e.
$Q_0 = C(\mathbf{0}, 1)$
$Q_1 = C\left((\mathbf{1}-\beta)\boldsymbol{a}, \beta\right)$
⋮

$Q_0 \setminus Q_1$ is not convex ⇒ i

$$\mathrm{Vol}(Q_0 \setminus Q_1) = \mathrm{Vol}(Q_0) -$$
$$= \frac{2^n}{n!} - \mathrm{Vol}($$

$Q_0 \cap Q_1$ is convex ⇒ tractable

We show that
if $1-\beta$ is sufficiently large,
i.e., $Q_0 \setminus Q_1$ is sufficiently large,
then
$$\mathrm{Vol}(Q_0 \setminus Q_1) \simeq \frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)$$
holds in the sense of approximation

Requirements to β are conflict, but is possible to be settled

Lemma 1

If $1-\beta \le \frac{c_1\epsilon}{n\|\boldsymbol{a}\|_1}$ where $0 < c_1\epsilon < 1$, then

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$$\mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) = \frac{1}{1-\beta^n}\left(\frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)\right)$$

Lemma 2

Suppose $1-\beta \ge \frac{c_2\epsilon}{n\|\boldsymbol{a}\|_1}$ where $0 < c_2\epsilon < 1$.

If we have Z approximating $\mathrm{Vol}(Q_0 \cap Q_1)$ such that

$$\mathrm{Vol}(Q_0 \cap Q_1) \le Z \le (1+\mathrm{c}_2\epsilon)\mathrm{Vol}(Q_0 \cap Q_1),$$

then $\frac{2^n}{n!} - Z$ satisfies

$$(1-\epsilon)\cdot\left(\frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)\right) \le \left(\frac{2^n}{n!} - Z\right) \le \left(\frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)\right)$$

Talk sketch

0. Introduction
1. Randomized approximation of counting
2. Deterministic approximation of volume I
3. Deterministic approximation of volume II

➢ FPTAS for the Volume of some $\mathcal{V}$-polytope

i. Problem description

ii. Idea: Reduction to $\mathrm{Vol}\big(C(0,1) \cap C(c,r)\big)$

iii. FPTAS for $\mathbf{Vol}\big(\boldsymbol{C}(\mathbf{0},\mathbf{1}) \cap \boldsymbol{C}(\boldsymbol{c},\boldsymbol{r})\big)$

iv. #P-hardness of $\mathrm{Vol}\big(C(0,1) \cap C(c,r)\big)$

The volume of an intersection of two cross-polytopes

Input: $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbb{Q}^n_{\geq 0}$ and r $(0 < r \leq 1)$ such that $\|\boldsymbol{c}\|_1 \leq r$.

Output: approximation of $\mathrm{Vol}(C(\mathbf{0}, 1) \cap C(\boldsymbol{c}, r))$

Recall

$$C(\boldsymbol{c}, r) \stackrel{\text{def}}{=} \mathrm{conv}\{\, \boldsymbol{c} \pm r\boldsymbol{e_i} \mid i = 1, \ldots n \,\}$$
$$= \{\, \boldsymbol{c} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{c}\|_1 \leq r \,\}$$

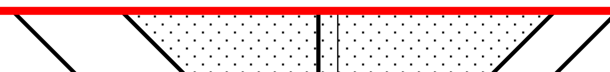
Lemma [Ando & Kijima 16+]

Suppose $\|\boldsymbol{c}\|_1 \leq r$. For any δ $(0 < \delta < 1)$, let $M \coloneqq \lceil 4n^2\delta^{-1} \rceil$

then $G_n(1, r)$ satisfies

$$\mathrm{Vol}\big(C(\mathbf{0}, 1) \cap C(\boldsymbol{c}, r)\big) \leq G_n(1, r) \leq (1 + \delta) \cdot \mathrm{Vol}\big(C(\mathbf{0}, 1) \cap C(\boldsymbol{c}, r)\big).$$

$G_n(1, r)$ is calculated in $O(n^7\delta^{-3})$ time.



Algorithm is based on an approximate convolution

Convolution to compute $\mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c},r))$

Def.

Let

$$\Psi_0 = \begin{cases} 1 & \text{if } u \geq 0 \text{ and } v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and recursively (w.r.t. i), let

$$\Psi_i(u,v) \stackrel{\text{def}}{=} \int_{-1}^{1} \Psi_{i-1}(u-|s|, v-|s-c_i|)\mathrm{d}s$$

Prop.

$$\Psi_n(1,r) = \mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c},r))$$

Proof sketch.

$$\frac{1}{2^i}\Psi_i(u,v) = \Pr\left[\left(\sum_{j=1}^{i} |X_j| \leq u\right) \wedge \left(\sum_{j=1}^{i} |X_j - c_j| \leq v\right)\right]$$

Riemann sum

Def.

Let

$$G_0 \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } u \geq 0 \text{ and } v \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Recursively, let

$$\overline{G_i}(u,v) \stackrel{\text{def}}{=} \int_{-1}^{1} G_{i-1}(u - |s|, v - |s - c_i|)\mathrm{d}s$$

and let

$$G_i(u,v) \stackrel{\text{def}}{=} \overline{G_i}\left(\frac{1}{M}\lceil Mu \rceil, \frac{r}{M}\left\lceil \frac{M}{r} v \right\rceil\right)$$

Notice that $G_i(u,v)$ is a step function, which implies that $\int$ appearing in the def. of $\overline{G_i}(u,v)$ is replaced by $\sum$.

G_i is calculated efficiently (omit the detail)

Let

$$\overline{G_i}(u,v) = \int_{-1}^{1} G_{i-1}(u-|s|, v-|s-c_i|)\mathrm{d}s$$
$$= \cdots$$
$$= \sum_{j=0}^{m-1} \left(t_{j+1}-t_j\right) \cdot G_{i-1}\left(\frac{1}{M}\left\lceil M\left(u-\left|t_{j+1}\right|\right)\right\rceil, \frac{r}{M}\left\lceil \frac{M}{r}\left(v-\left|t_{j+1}-c_i\right|\right)\right\rceil\right)$$

where $t_0, t_1, \dots, t_m$ ($m = O(M)$) are event points.

⇨ meaning that we can compute G_i (without $\int$)

The volume of an intersection of two cross-polytopes

Input: $\boldsymbol{c} = (c_1, \dots, c_n) \in \mathbb{Q}^n_{\geq 0}$ and r $(0 < r \leq 1)$ such that $\|\boldsymbol{c}\|_1 \leq r$.

Output: approximation of $\mathrm{Vol}(C(\boldsymbol{0}, 1) \cap C(\boldsymbol{c}, r))$

Recall

$$C(\boldsymbol{c}, r) \stackrel{\text{def}}{=} \mathrm{conv}\{\boldsymbol{c} \pm r\boldsymbol{e_i} \mid i = 1, \dots n\}$$
$$= \{\boldsymbol{c} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{c}\|_1 \leq r\}$$

Lemma [Ando & Kijima 16+]

Suppose $\|\boldsymbol{c}\|_1 \leq r$. For any δ $(0 < \delta < 1)$, let $M := \lceil 4n^2\delta^{-1} \rceil$

then $G_n(1, r)$ satisfies

$$\mathrm{Vol}\big(C(\boldsymbol{0}, 1) \cap C(\boldsymbol{c}, r)\big) \leq G_n(1, r) \leq (1 + \delta) \cdot \mathrm{Vol}\big(C(\boldsymbol{0}, 1) \cap C(\boldsymbol{c}, r)\big).$$

$G_n(1, r)$ is calculated in $O(n^7\delta^{-3})$ time.

Analysis of approximation ratio uses the techniques
- ✓ "horizontal approximation" and
- ✓ "cone bound"

in a similar way as 0-1 knapsack

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iv. #P-hardness of $\mathbf{Vol}\big(\boldsymbol{C}(\mathbf{0},\mathbf{1}) \cap \boldsymbol{C}(\boldsymbol{c},\boldsymbol{r})\big)$

Volume of an intersection of two L_1 balls is #P-hard

Thm.

Let $\boldsymbol{c} = (c_1, \dots, c_n) \in \mathbb{Z}_{\geq 0}^n$ and $r_1, r_2 \in \mathbb{Z}$,

such that $\|\boldsymbol{c}\|_1 \leq \min(r_1, r_2)$, i.e., $\boldsymbol{c} \in C(\boldsymbol{0}, r_1)$ and $\boldsymbol{0} \in C(\boldsymbol{c}, r_2)$

Then computing $\mathrm{Vol}\big(C(\boldsymbol{0}, r_1) \cap C(\boldsymbol{c}, r_2)\big)$ is #P-hard.

Proof sketch

Reduce (a version of) counting subset sum.

Intuitively, as given $\boldsymbol{a} \in \mathbb{Z}_{>0}^n$, we show that

$$\mathrm{Vol}\big(C(\boldsymbol{0}, 1+\epsilon) \cap C(\delta\boldsymbol{a}, 1)\big) - \mathrm{Vol}\big(C(\boldsymbol{0}, 1) \cap C(\delta\boldsymbol{a}, 1)\big)$$

$$\simeq \frac{\epsilon}{n!} \left| \boldsymbol{\sigma} \in \{-1,1\}^n \mid \textstyle\sum_{i=1}^n \sigma_i a_i > 0 \right|$$

when $0 < \epsilon < \delta \ll \frac{1}{\|\boldsymbol{a}\|_1}$.

Volume of an intersection of two L_1 balls is #P-hard

Claim

$$\mathrm{Vol}\big(C(\mathbf{0},1+\epsilon)\cap C(\delta\boldsymbol{a},1)\big)-\mathrm{Vol}\big(C(\mathbf{0},1)\cap C(\delta\boldsymbol{a},1)\big)\simeq\frac{\epsilon}{n!}\,|\,\boldsymbol{\sigma}\in\{-1,1\}^n\mid\textstyle\sum_{i=1}^n\sigma_i a_i>0\,|$$

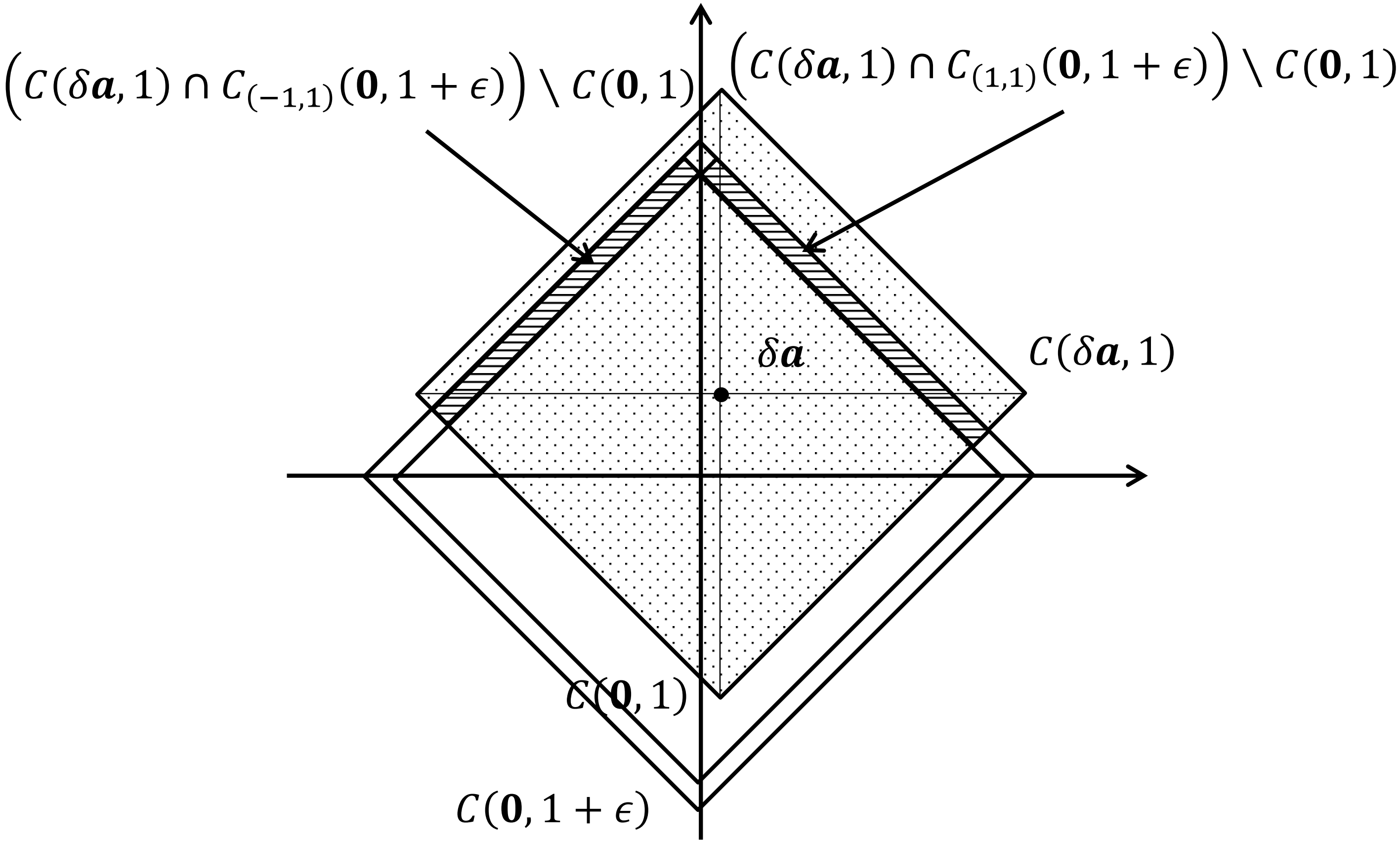

Fig. When an instance does not have a subset-sum solution

Volume of an intersection of two L_1 balls is #P-hard

Claim

$$\mathrm{Vol}\big(C(\mathbf{0},1+\epsilon)\cap C(\delta\boldsymbol{a},1)\big)-\mathrm{Vol}\big(C(\mathbf{0},1)\cap C(\delta\boldsymbol{a},1)\big)\simeq\frac{\epsilon}{n!}\left|\,\boldsymbol{\sigma}\in\{-1,1\}^n \mid \textstyle\sum_{i=1}^{n}\sigma_i a_i>0\,\right|$$

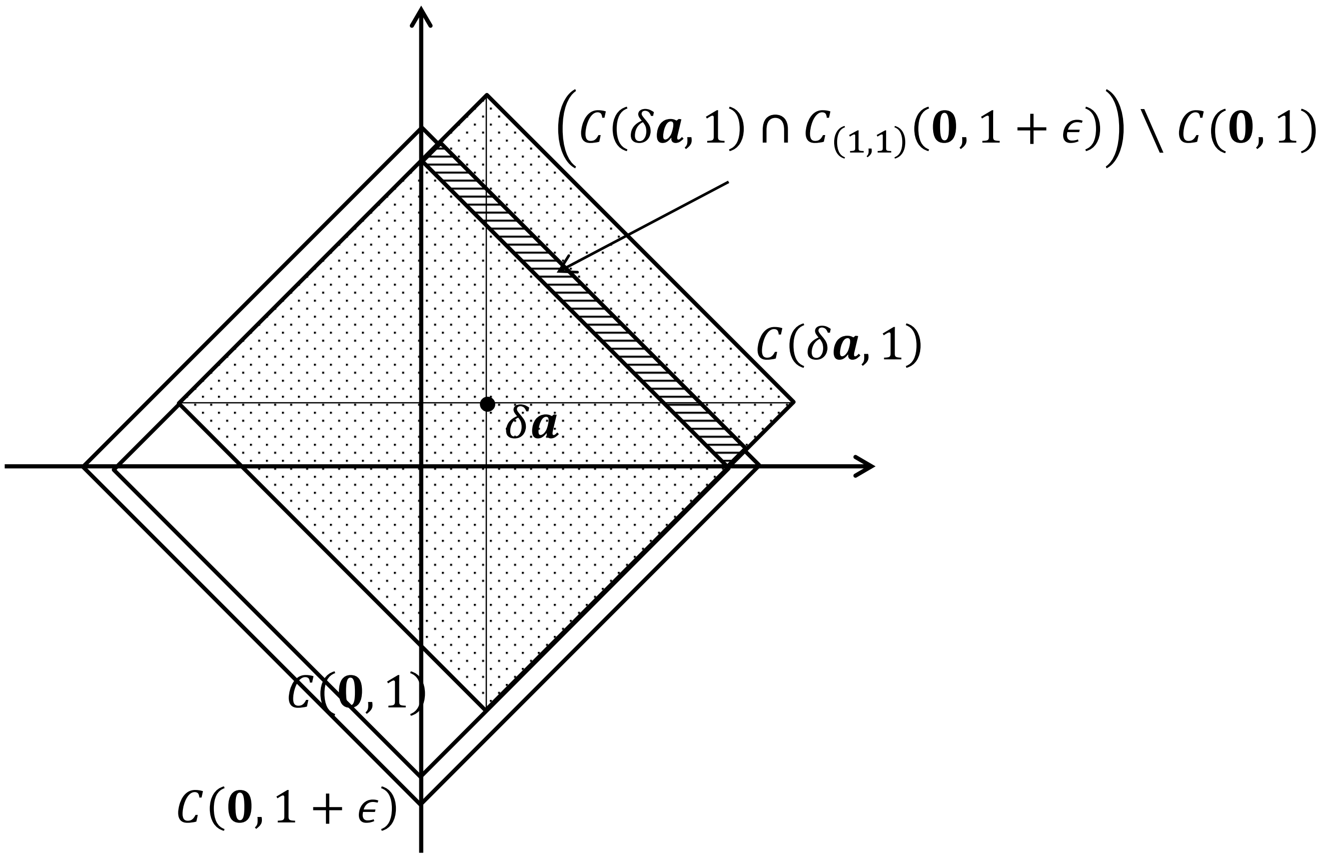

Fig. When an instance has a subset-sum solution

Open problems

V-polytope

- When is computing the n-D volume of a $\mathcal{V}$-polytope hard?
  - ✓ It is #P-hard for at most $2n+1$ vertices [Khachiyan]
  - ✓ It is $\mathrm{poly}(n)$ time (using some $\sqrt{}$ s) for n+const. vertices.
    - e.g., $n+1$ vertices implies simplex, which is easily computed.
- Is there an FPTAS for any V-polytope?
- What is known about the volume for "duality" of polytopes
- How many vertices of intersection of two cross-polytopes?

Deterministic approximation for #P-hard problems

- #linear extensions ?

Concluding Remarks

Talk sketch

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Randomness is necessary for some computation.

But when?

The end

Thank you for the attention.