Rapidly Mixing Chain and Perfect Sampler for Logarithmic Separable Concave Distributions on Simplex

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Problem

Given

each f_i is log-concave

log-concave functions vector $\boldsymbol{f} = (f_1, \ldots, f_n)$

Sample space

simplex

$$\Xi = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_i \geq 0,\ x_1 + \cdots + x_n = K\}$$

Probability function

logarithmic separable concave function

$$\pi(\boldsymbol{x}) = \frac{1}{C} \prod_{i=1}^{n} f_i(x_i)$$

where $C \stackrel{\text{def.}}{=} \sum_{\boldsymbol{x} \in \Xi} \prod_{i=1}^{n} f_i(x_i)$

log-concave function f_i

$f_i : \mathbb{Z} \to \mathbb{R}_{++}$ ($i \in \{1, \ldots, n\}$),

$2 \ln f_i (z+1) \geq \ln f_i (z) + \ln f_i (z+2)$.

ex. of log-concave
- exponential dist.,
- normal dist.,
- positive concave.

logarithmic separable concave function

product of single variable log-concave functions.

Main results

For any logarithmic separable concave discrete distribution on simplex, we give two hit-and-run chains.

- One provides an approximate sampler.
  - The mixing time $\tau(\varepsilon)$ satisfies

$$\tau(\varepsilon) \leq \frac{n(n-1)}{2} \ln(K\varepsilon^{-1}).$$

  - Proof by path coupling.
- The other provides a perfect sampler.
  - The chain is monotone.
  - The perfect sampler is based on monotone CFTP (Coupling From The Past)

$\Rightarrow$ Exactly according to the stationary distribution.

Related works and applications

Related works

- A. Frieze and R. Kannan ('97)
  - ➢ log-concave on continuous polytope
  - $\Rightarrow$ polynomial time mixing
- L. Lovasz and S. Vempara ('02)
  - ➢ log-concave on continuous polytope
  - $\Rightarrow$ mixing time is $O^*(n^4)$
- D. Randall and P. Winker ('05)
  - ➢ uniform on continuous simplex
  - $\Rightarrow$ mixing time is $\Theta(n^3 \ln n)$

Discrete is NOT easy

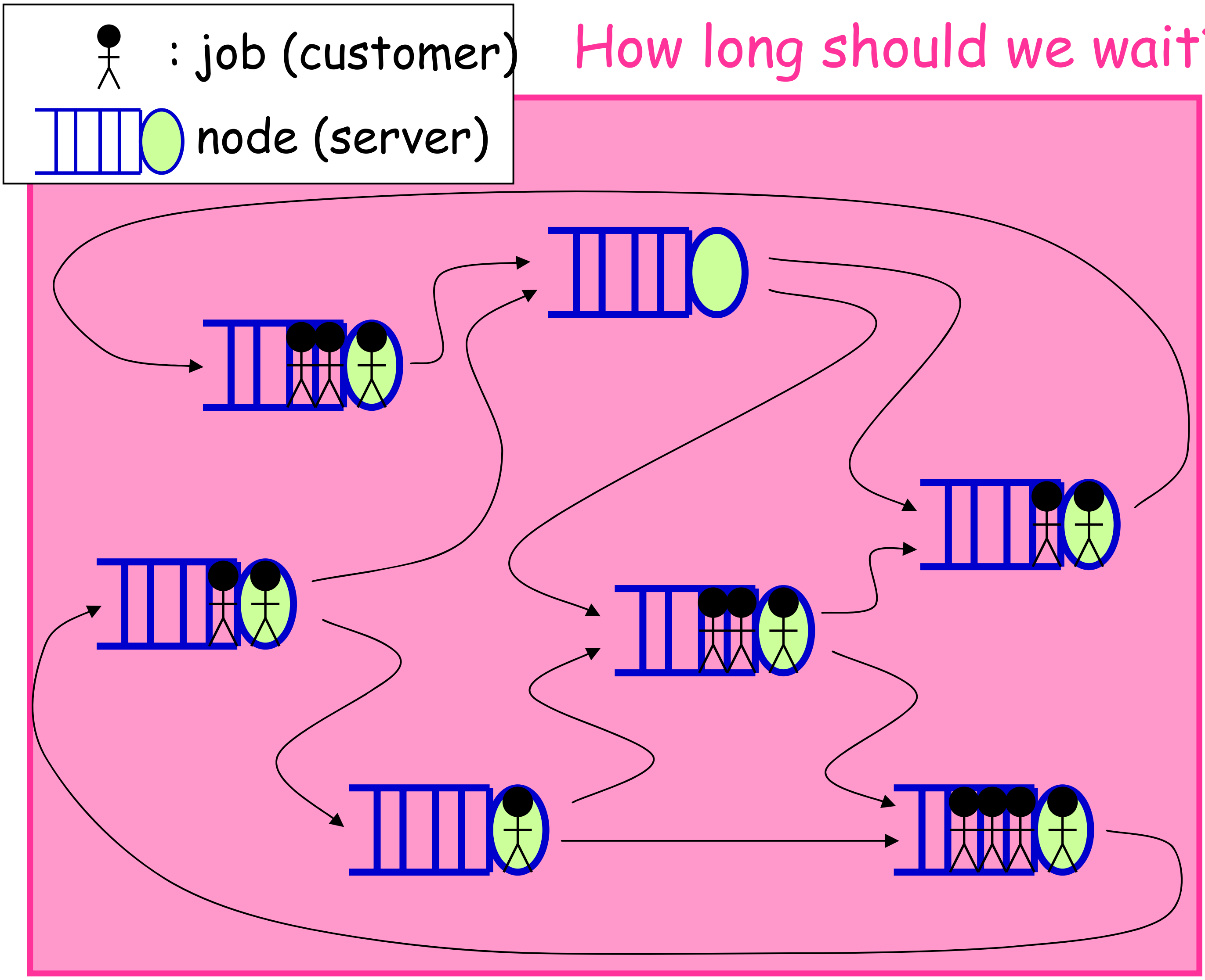
Applications

- Jackson networks
- Discretized Dirichlet Distribution

An application

- Jackson networks: Queueing network theory

How long should we wait?



Steady state distribution of jobs

$$\pi_{\mathrm{J}}(\boldsymbol{x}) \stackrel{\text{def.}}{=} \frac{1}{C_{\mathrm{J}}} \prod_{i=1}^{n} \frac{1}{\prod_{j=1}^{x_i} \min\{j, s_i\}} \left(\frac{\theta_i}{\mu_i}\right)^{x_i}$$

with given positive constant vectors

$(\theta_1\theta_2,\ldots,\theta_n)$, $(\mu_1,\mu_2,\ldots,\mu_n)$ and $(s_1s_2,\ldots,s_n)$

Another application

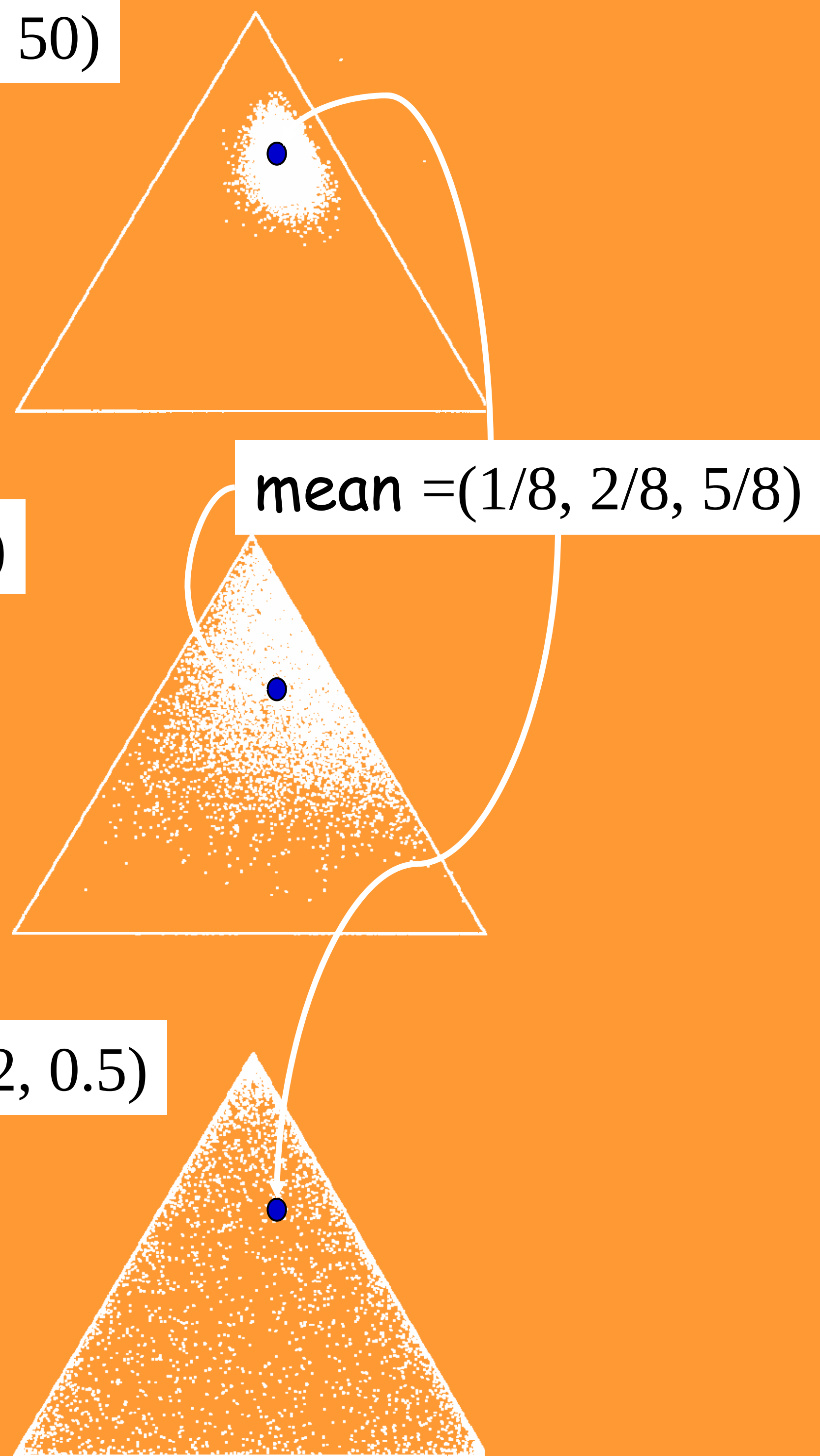
- Discretized Dirichlet Distribution

$$\pi_{\mathrm{D}} \stackrel{\text{def.}}{=} \frac{1}{C_{\mathrm{D}}} \prod_{i=1}^{n} \left(\frac{x_i}{K}\right)^{u_i - 1}$$

$(u_1, u_2, u_3) = (10, 20, 50)$

$(u_1, u_2, u_3) = (1, 2, 5)$

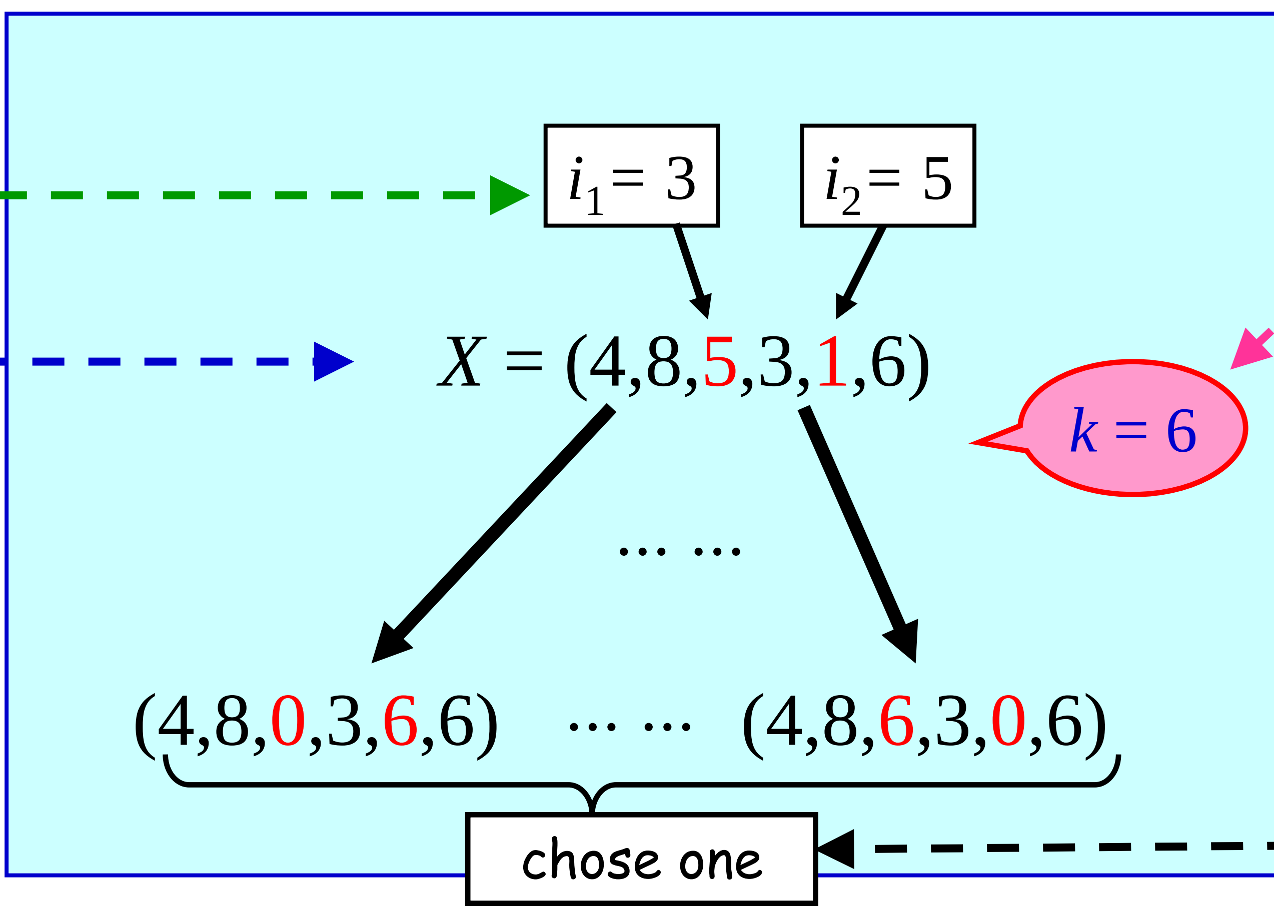
$(u_1, u_2, u_3) = (0.1, 0.2, 0.5)$



Markov chain

Current state: $X \in \Xi$

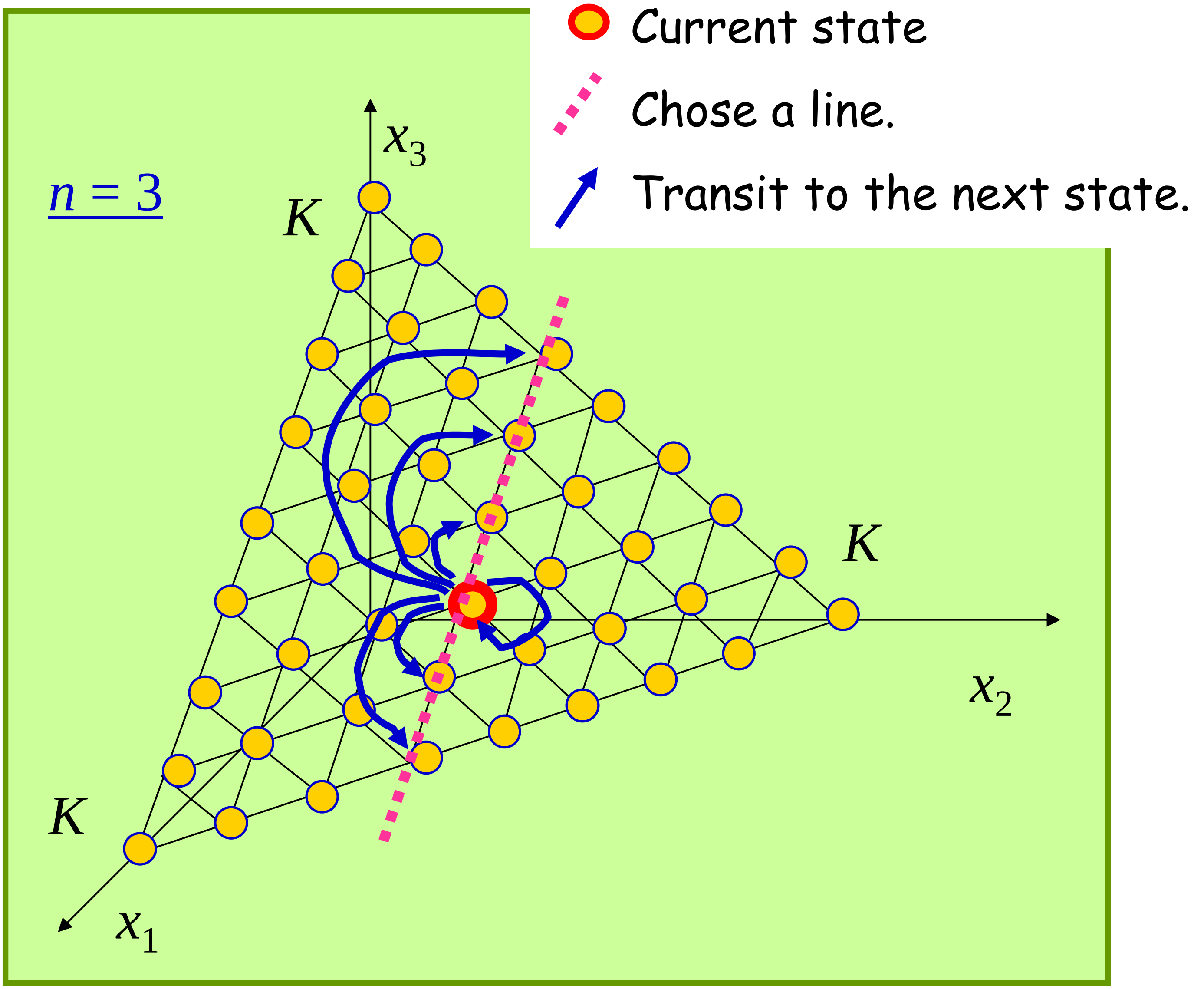
- Chose a pair of indices $\{i_1, i_2\}$, u. a. r.
- Set $k = X_{i_1} + X_{i_2}$
- Chose $l \in \{0, 1, \ldots, k\}$ w. p. $\dfrac{f_{i_1}(l) f_{i_2}(k-l)}{\sum_{s=0}^{k} f_{i_1}(s) f_{i_2}(k-s)}$
- Set $X'_i = \begin{cases} l & (i = i_1) \\ k - l & (i = i_2) \\ X_i & (\text{otherwise}) \end{cases}$



Stationary distribution

Them.

The chain is ergodic and has a unique stationary distribution which is π



Approximate sampler

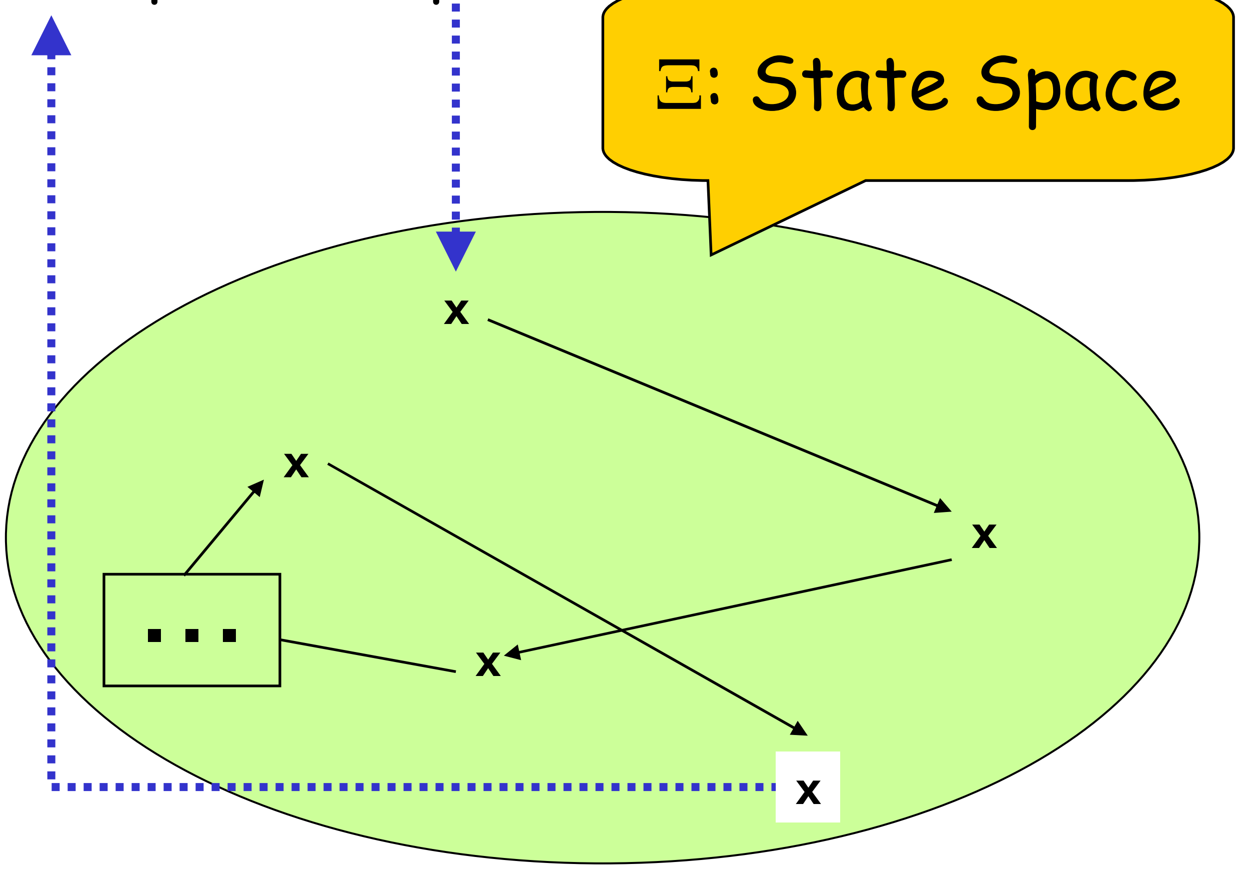
- Sampling via Markov chain

Basic idea

1: Start from an initial state

2: Make several transitions

3: Output a sample



If a chain is ergodic, it converges to a unique stationary distribution.

Mixing time

Them. 1

The mixing time $\tau(\varepsilon)$ of the chain satisfies

$$\tau(\varepsilon) \leq \frac{n(n-1)}{2} \ln(K\varepsilon^{-1}).$$

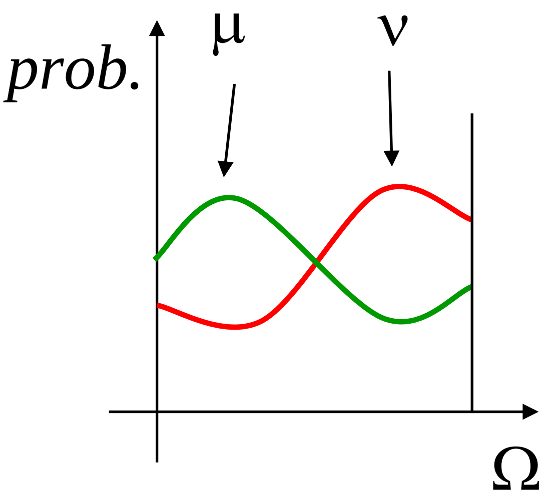
Total variation distance between μ and ν is defined by

$$d_{\mathrm{TV}}(\mu,\nu) \overset{\text{def.}}{=} \max_{Q} \left\{ \sum_{x\in Q} (\mu(x) - \nu(x)) \right\}$$

$$\equiv \frac{1}{2} \sum_{x\in Q} |\mu(x) - \nu(x)|$$

Mixing time of a chain is defined by

$$\tau(\varepsilon) \overset{\text{def.}}{=} \max_{x\in\Omega} \{\min\{t \mid \forall s \geq t,\ d_{\mathrm{TV}}(P_x^t, \pi) \leq \varepsilon\}\}$$



Proof (by path coupling)

- Graph: $G = (\Xi, E)$
- Edge: $\{X,Y\} \in E$

$$\Leftrightarrow \exists\{j_1, j_2\}\ (j_1 \neq j_2),\ X_j - Y_j = \begin{cases} 1 & (j \in \{j_1, j_2\}), \\ 0 & (\text{otherwise}) \end{cases}$$

- Distance: $d_A(X,Y)$ shortest path between X and Y on G
- Joint process: (X,Y) |→ (X',Y') cumulative dist.

Claim

$$\forall\{X,Y\} \in E,\ \mathrm{E}[d_A(X',Y')] \leq \left(1 - \frac{2}{n(n-1)}\right) d_A(X,Y)$$

Proof: Let $\{X, Y\} \in E$ and $\{j_1, j_2\}$ $(j_1 \neq j_2)$ be a supporting pair. Suppose a pair $\{i_1, i_2\}$ $(i_1 \neq i_2)$ is chosen.

Case 1. $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$ (neither of j_1, j_2 are chosen)

- $d_A(X', Y') = d_A(X, Y) \equiv 1$

Case 2. $\{i_1, i_2\} = \{j_1, j_2\}$ (both of j_1, j_2 are chosen)

- $d_A(X', Y') = 0$ ⟵ w. p. $\frac{2}{n(n-1)}$

Case 3. $|\{i_1, i_2\} \cap \{j_1, j_2\}| = 1$ (exactly one of j_1, j_2 is chosen)

- $d_A(X', Y') = d_A(X, Y) \equiv 1$

Them. Path coupling [Bubley and Dyer '97]

$0 \leq \exists\beta < 1, \forall\{X, Y\} \in E,\ \mathrm{E}[d(X', Y')] \leq \beta \cdot d(X, Y)$

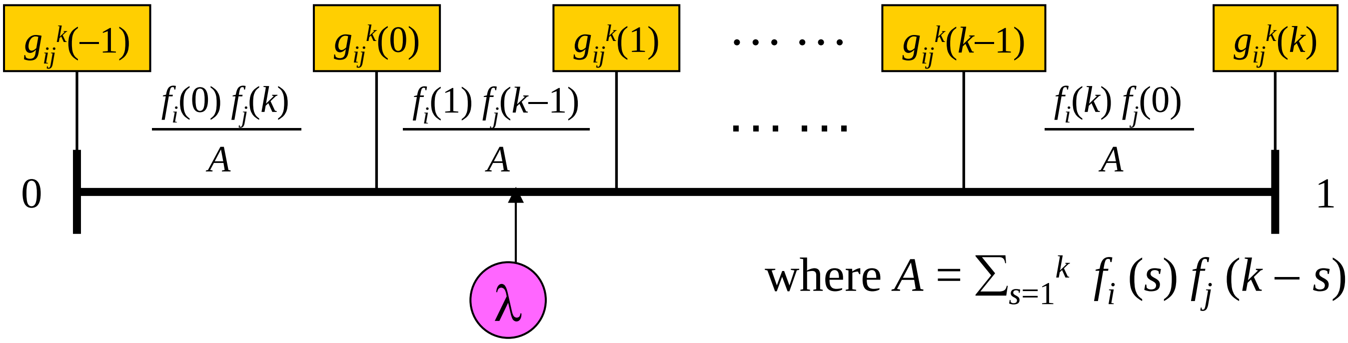
$\Rightarrow \tau(\varepsilon) \leq (1-\beta)^{-1} \ln(D\varepsilon^{-1})$. ■

Alternating inequalities

Joint process

Simulate the chain with random number $\lambda \in [0,1)$ and cumulative distribution function

$$g_{ij}^{k}(l) \stackrel{\text{def.}}{=} \frac{\sum_{s=1}^{l} f_i(s) f_j(k-s)}{\sum_{s=1}^{k} f_i(s) f_j(k-s)}$$



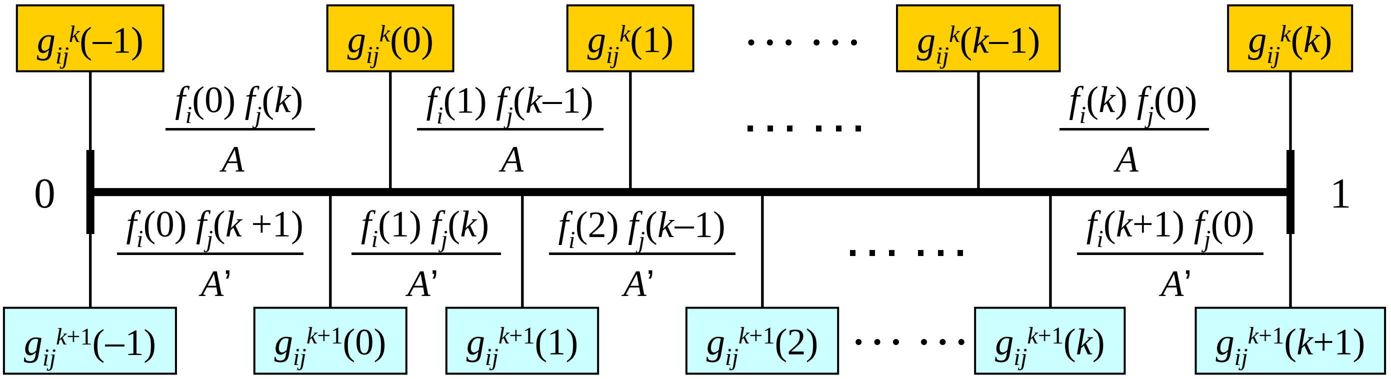
where $A = \sum_{s=1}^{k} f_i(s) f_j(k-s)$.

Lemma

A logarithmic separable concave function satisfies alternating inequalities

$$g_{ij}^{k+1}(l) \leq g_{ij}^{k}(l) \leq g_{ij}^{k+1}(l+1) \; \forall l \in \{0, 1, \ldots, k\}$$

for any $k \in \{0, \ldots, K\}$ and for any pair $\{i, j\}$.



Another chain - for perfect sampling

modification

consecutive

- Chose a pair of indices $\{i, i+1\}$, u. a. r.

Them. 2

The chain is monotone.

Equipments

- Cumulative sum vector c_x for $x \in \Xi$

$$c_{\boldsymbol{x}}(i) \stackrel{\text{def.}}{=} \begin{cases} 0 & (i = 0) \\ \sum_{j=1}^{i} x_j & (i \in \{1, 2, \ldots, n\}) \end{cases}$$

- Partial order on Ξ

$$\boldsymbol{x} \succeq \boldsymbol{y} \ (\boldsymbol{x}, \boldsymbol{y} \in \Xi) \stackrel{\text{def.}}{\Leftrightarrow} c_{\boldsymbol{x}} \geq c_{\boldsymbol{y}} \ (\forall i \in \{0, 1, \ldots, n\})$$

- Max. and min.

Max.: $x_U = (K, 0, 0, \ldots, 0, 0)$

Min.: $x_L = (0, 0, 0, \ldots, 0, K)$

- update function

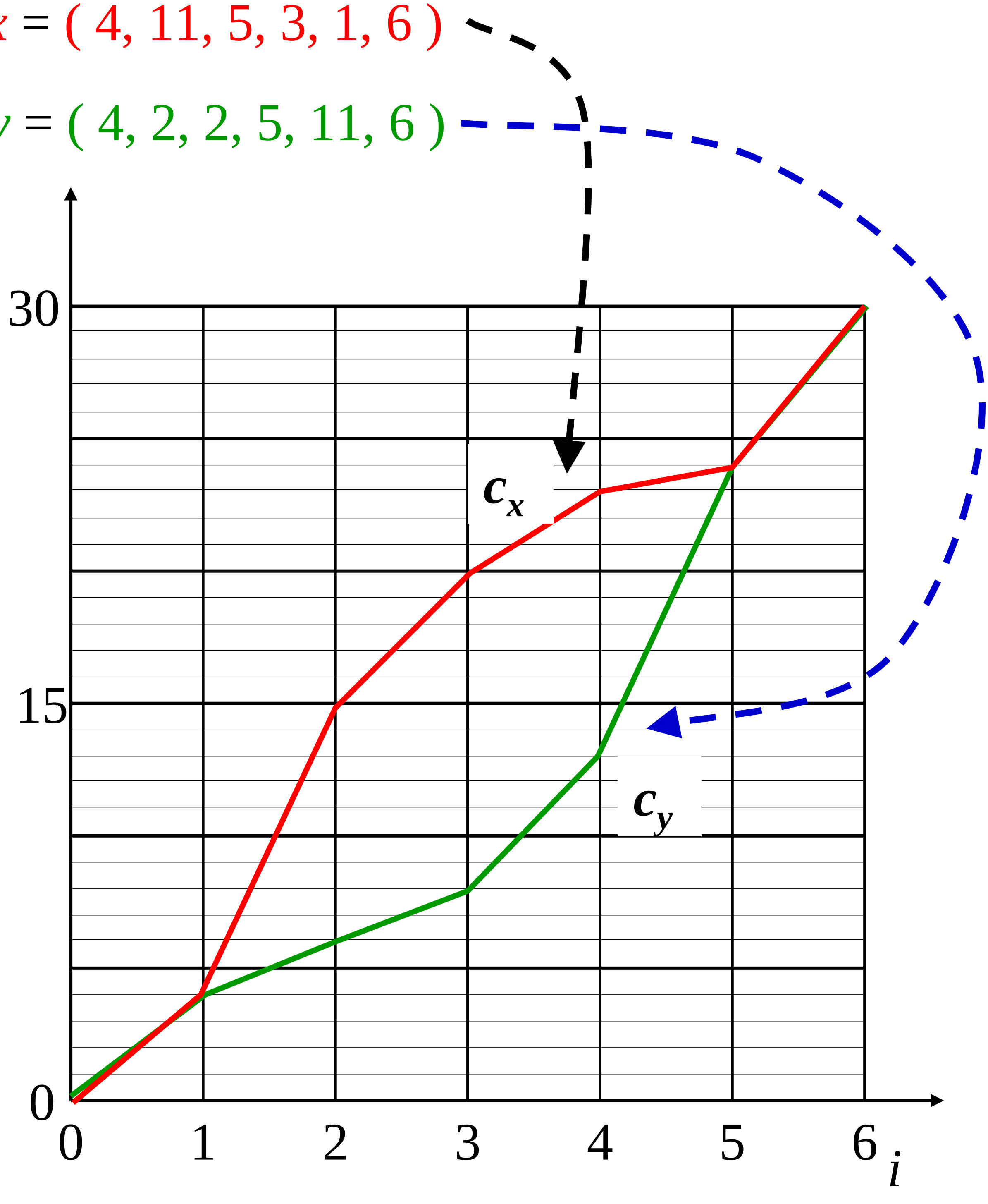
cumulative distribution function

Cumulative sum vector

Example

$x = (4, 11, 5, 3, 1, 6)$

$y = (4, 2, 2, 5, 11, 6)$



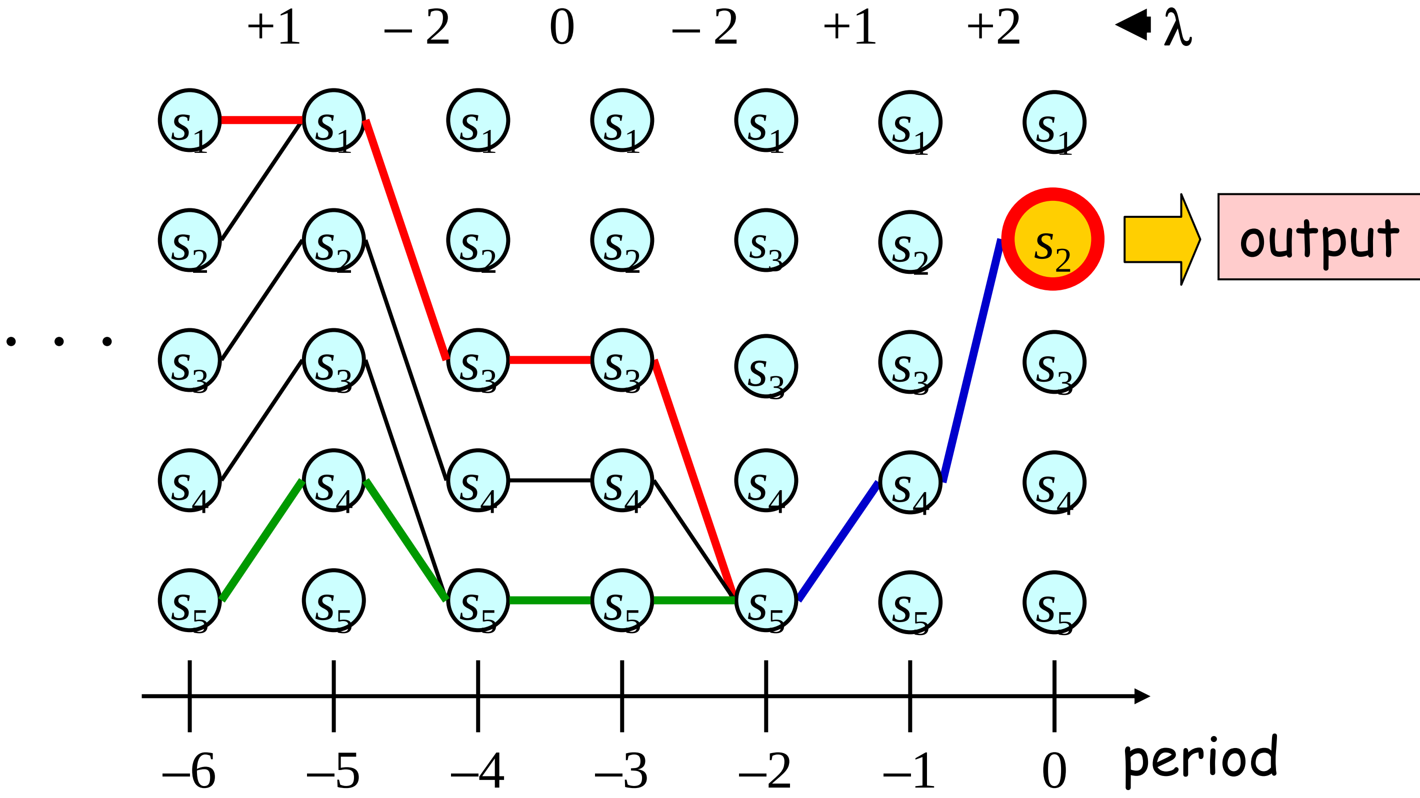
Thus $x \succeq y$

monotone CFTP [Propp and Wilson '96]

monotone Markov chain

- The state space of a chain has a poset.
- Any transitions preserve partial order
- The poset has a unique max and min.

monotone CFTP



Them. monotone CFTP

monotone CFTP algorithm returns a sample in probabilistic finite time, exactly according to the stationary distribution.

Concluding remarks

Future works

- Is the mixing time (or expected coalescence time) of our monotone chain bounded by polynomial time?
- Extension to log-concave (not separable)
  - application: universal portfolio.
- Extension to integer points on more general polytopes?
  - application: loss networks.
  - base polytope of submodular function.

Technical reports are available from

http://www.keisu.t.u-tokyo.ac.jp/Research/techrep.0.html