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10. multi-commodity flow (多品種流)

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マルコフ連鎖の関数解析

Laplacian matrix: $L = I - P$

- λ がPの固有値 $\Leftrightarrow 1 - \lambda$が$L$の固有値．固有ベクトルは共通．

  $\because\ P\boldsymbol{v} = \lambda\boldsymbol{v}$ とすると，$L\boldsymbol{v} = (I - P)\boldsymbol{v} = \boldsymbol{v} - P\boldsymbol{v} = \boldsymbol{v} - \lambda\boldsymbol{v} = (1 - \lambda)\boldsymbol{v}$.

- Lの固有値は0以上2以下．

  $\because\ -1 \leq \lambda \leq 1$より$0 \leq 1 - \lambda \leq 2$

内積 $\langle\varphi, \psi\rangle = \sum_{x\in\Omega} \varphi(x)\psi(x)$

π内積 $\langle\varphi, \psi\rangle_\pi = \sum_{x\in\Omega} \pi(x)\varphi(x)\psi(x) \quad (= \mathbb{E}_\pi[\varphi\psi])$

Dirichlet form

$$\mathcal{E}_P(\varphi, \varphi) = \langle\varphi, L\varphi\rangle_\pi = \langle\varphi, (I - P)\varphi\rangle_\pi = \sum_{x\in\Omega} \pi(x)\varphi(x) \sum_{y\in\Omega} \big(I(x,y) - P(x,y)\big)\varphi(y)$$

定理 10.1.

Pがエルゴード的でreversibleのとき，

$$1 - \lambda_2 = \inf_{\varphi \not\equiv c} \frac{\mathcal{E}_P(\varphi, \varphi)}{\mathrm{Var}_\pi[\varphi]}$$

$$\mathrm{Var}_\pi[\varphi] = \mathbb{E}_\pi[(\varphi - \mathbb{E}_\pi[\varphi])^2] = \sum_{x\in\Omega} \pi(x)(\varphi(x) - \mathbb{E}_\pi[\varphi])^2$$

Intuition: Pが対称の場合

命題 10.2.

Pがエルゴード的で対称の時,

$$1-\lambda_2 = \inf_{\sum\varphi(x)=0,\varphi\neq 0} \frac{\mathcal{E}_P(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_\pi}$$

証明:

- P対称のとき，$\pi(x)=\frac{1}{|\Omega|}$.
- $\frac{\mathcal{E}_P(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_\pi} = \frac{\langle\varphi,L\varphi\rangle_\pi}{\langle\varphi,\varphi\rangle_\pi} = \frac{\langle\varphi,L\varphi\rangle}{\langle\varphi,\varphi\rangle}$
- $\psi_1=\pi\propto 1$なので

$$\inf_{\varphi\perp 1,\varphi\neq 0}\frac{\langle\varphi,L\varphi\rangle}{\langle\varphi,\varphi\rangle} = 1-\lambda_2$$

- $\varphi\perp 1 \Leftrightarrow \sum_x\varphi(x)=0$に注意.

Rayleigh商

ψ_iを$1-\lambda_i$の固有関数として,

$\varphi=\sum_{i=1}^n c_i\,\psi_i$とすると,

$$\frac{\langle\varphi,L\varphi\rangle}{\langle\varphi,\varphi\rangle} = \frac{\sum_{i=1}^n(1-\lambda_i)c_i^2\langle\psi_i,\psi_i\rangle}{\sum_{i=1}^n a_i^2\langle\psi_i,\psi_i\rangle} \geq \frac{\min(1-\lambda_i)\sum_{i=1}^n c_i^2\langle\psi_i,\psi_i\rangle}{\sum_{i=1}^n c_i^2\langle\psi_i,\psi_i\rangle} = \min(1-\lambda_i)$$

$\varphi\perp 1,\varphi\not\equiv 0$のとき$\min(1-\lambda_i)=1-\lambda_2$

定理10.1.の証明

$\varphi' = \Pi^{1/2}\varphi$

証明:

$$\frac{\mathcal{E}_P(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_\pi} = \frac{\langle\varphi, L\varphi\rangle_\pi}{\langle\varphi,\varphi\rangle_\pi} = \frac{\langle\Pi^{1/2}\varphi, \Pi^{1/2}L\varphi\rangle}{\langle\Pi^{1/2}\varphi, \Pi^{1/2}\varphi\rangle} = \frac{\langle\varphi,' \ \Pi^{1/2}L\Pi^{-1/2}\varphi'\rangle}{\langle\varphi',\varphi'\rangle} = \frac{\langle\varphi,' \ A\varphi'\rangle}{\langle\varphi',\varphi'\rangle}$$

- $\psi_1' = \sqrt{\pi}$なので

$$\inf_{\varphi'\perp\sqrt{\pi}\ \varphi'\not\equiv 0} \frac{\langle\varphi,' \ A\varphi'\rangle}{\langle\varphi',\varphi'\rangle} = 1 - \lambda_2$$

$A = \Pi^{1/2}L\Pi^{-1/2}$の固有値0の固有ベクトル $(\pi_1^{1/2}, \ldots, \pi_N^{1/2})$ (第9回)

- $\varphi' \perp \sqrt{\pi} \Leftrightarrow \langle\varphi,' \sqrt{\pi}\rangle = 0 \Leftrightarrow \langle\Pi^{1/2}\varphi, \Pi^{1/2}1\rangle = 0 \Leftrightarrow \langle\varphi, 1\rangle_\pi = 0$に注意.

$$\inf_{\varphi\perp 1, \varphi\not\equiv 0} \frac{\langle\varphi, L\varphi\rangle_\pi}{\langle\varphi,\varphi\rangle_\pi} = \inf_{\varphi\perp 1, \varphi\not\equiv 0} \frac{\langle\varphi, L\varphi\rangle_\pi}{\langle\varphi,\varphi\rangle_\pi - \langle\varphi,1\rangle_\pi^2}$$

$\longleftarrow$ $\varphi \perp 1 \Rightarrow \langle\varphi, 1\rangle_\pi = 0$

$$= \inf_{\varphi\not\parallel 1, \varphi\not\equiv 0} \frac{\langle\varphi, L\varphi\rangle_\pi}{\langle\varphi,\varphi\rangle_\pi - \langle\varphi,1\rangle_\pi^2}$$

$$= \inf_{\varphi\not\equiv c} \frac{\langle\varphi, L\varphi\rangle_\pi}{\langle\varphi,\varphi\rangle_\pi - \langle\varphi,1\rangle_\pi^2}$$

$$= \inf_{\varphi\not\equiv c} \frac{\mathcal{E}_P(\varphi,\varphi)}{\mathrm{Var}_\pi[\varphi]}$$

$$\begin{aligned}\mathcal{E}_P(\varphi,\varphi) &= \langle\varphi, L\varphi\rangle_\pi \\ \mathrm{Var}_\pi[\varphi] &= \mathbb{E}_\pi[(\varphi - \mathbb{E}_\pi[\varphi])^2] \\ &= \mathbb{E}_\pi[\varphi^2] - \mathbb{E}_\pi[\varphi]^2 \\ &= \langle\varphi,\varphi\rangle_\pi - \langle\varphi,1\rangle_\pi^2\end{aligned}$$

MULTI COMMODITY FLOW

- $Q(e) \coloneqq \pi(z)P(z,z')$: “capacity” (a.k.a. ergodic flow)
- $R(x,y) \coloneqq \pi(x)\pi(y)$: demand
- paths_{xy}: xからyの単純経路，$\mathrm{paths} = \bigcup_{xy} \mathrm{paths}_{xy}$
- f: paths $\to \mathbb{R}^+ \cup \{0\}$

$$\sum_{p \in \mathcal{P}_{xy}} f(p) = R(x,y) \quad \forall (x,y) \in \Omega \times \Omega$$

- $f(e) = \sum_{p \ni e} f(p)$
- $\rho(f) = \max_e \frac{f(e)}{Q(e)}$
- $D(f) = \max_{p : f(p) > 0} |p|$

定理 10.3.

Pがエルゴード的とする．任意のフローfに対して

$$\frac{1}{1-\lambda_2} \le \rho(f)D(f)$$

系 10.4. (cf. 定理8.1.)

$$\tau(\epsilon) = O\left(\rho(f)D(f)\left(\log \pi_{\min}^{-1} + \log \epsilon^{-1}\right)\right)$$

定理 10.3.

Pがエルゴード的とする．任意のフローfに対して

$$\frac{1}{1-\lambda_2} \leq \rho(f)D(f) \quad (*)$$

証明の方針

- 前の定理より$1-\lambda_2 = \inf_{\varphi \not\equiv c} \frac{\mathcal{E}_P(\varphi,\varphi)}{\mathrm{Var}_\pi[\varphi]}$.

$$(*) \Leftrightarrow \frac{1}{\inf_{\varphi \not\equiv c} \frac{\mathcal{E}_P(\varphi,\varphi)}{\mathrm{Var}_\pi[\varphi]}} \leq \rho(f)D(f) \Leftrightarrow \sup_{\varphi \not\equiv c} \frac{\mathrm{Var}_\pi[\varphi]}{\mathcal{E}_P(\varphi,\varphi)} \leq \rho(f)D(f)$$

- 任意の$\varphi \not\equiv c$に対して

$$\mathrm{Var}_\pi[\varphi] \leq \rho(f)D(f)\mathcal{E}_P(\varphi,\varphi)$$

を示す.

- $\mathrm{Var}_\pi[\varphi] = \frac{1}{2}\sum_{x,y} \pi(x)\pi(y)\big(\varphi(x)-\varphi(y)\big)^2$
- $\mathcal{E}_P(\varphi,\varphi) = \frac{1}{2}\sum_{x,y} \pi(x)P(x,y)\big(\varphi(x)-\varphi(y)\big)^2$

Note

$$
\begin{aligned}
\mathrm{Var}_\pi[\varphi] &= \mathbb{E}_\pi[(\varphi - \mathbb{E}_\pi[\varphi])^2] \\
&= \mathbb{E}_\pi[\varphi^2] - \mathbb{E}_\pi[\varphi]^2 \\
&= \sum_{x\in\Omega} \pi(x)\varphi(x)^2 - \left(\sum_{x\in\Omega} \pi(x)\varphi(x)\right)^2 \\
&= \sum_{x\in\Omega} \pi(x)\varphi(x)^2 \sum_{y\in\Omega} \pi(y) - \left(\sum_{x\in\Omega} \pi(x)\varphi(x)\right)^2 \\
&= \sum_{x\in\Omega}\sum_{y\in\Omega} \pi(x)\pi(y)\,\frac{\varphi(x)^2 + \varphi(y)^2}{2} - \left(\sum_{x\in\Omega} \pi(x)\varphi(x)\right)\left(\sum_{y\in\Omega} \pi(y)\varphi(y)\right) \\
&= \frac{1}{2}\sum_{x\in\Omega}\sum_{y\in\Omega} \pi(x)\pi(y)(\varphi(x)^2 + \varphi(y)^2) - \sum_{x\in\Omega}\sum_{y\in\Omega} \pi(x)\pi(y)\varphi(x)\varphi(y) \\
&= \frac{1}{2}\sum_{x,y} \pi(x)\pi(y)((\varphi(x)^2 + \varphi(y)^2) - 2\varphi(x)\varphi(y)) \\
&= \frac{1}{2}\sum_{x,y} \pi(x)\pi(y)\big(\varphi(x) - \varphi(y)\big)^2
\end{aligned}
$$

Note

$$
\begin{aligned}
\mathcal{E}_P(\varphi,\varphi) &= \langle \varphi, L\varphi\rangle_\pi = \langle \varphi, (I-P)\varphi\rangle_\pi \\
&= \sum_{x\in\Omega} \pi(x)\varphi(x) \sum_{y\in\Omega} \big(I(x,y)-P(x,y)\big)\varphi(y) \\
&= \sum_{x,y} \pi(x)\varphi(x)\big(I(x,y)-P(x,y)\big)\varphi(y) \\
&= \sum_{x,y} \pi(x)\varphi(x)I(x,y)\varphi(y) - \sum_{x,y} \pi(x)\varphi(x)P(x,y)\varphi(y) \\
&= \sum_{x} \pi(x)\varphi(x)^2 - \sum_{x,y} \pi(x)P(x,y)\varphi(x)\varphi(y) \\
&= \frac{1}{2}\sum_{x,y} \pi(x)\pi(y)(\varphi(x)^2+\varphi(y)^2) - \sum_{x,y} \pi(x)P(x,y)\varphi(x)\varphi(y) \\
&= \frac{1}{2}\sum_{x,y} \pi(x)P(x,y)\big(\varphi(x)-\varphi(y)\big)^2
\end{aligned}
$$

証明

$$2\mathrm{Var}_\pi[\varphi] = \sum_{xy} \pi(x)\pi(y)\big(\varphi(x)-\varphi(y)\big)^2$$

$$= \sum_{xy}\sum_{p\in\mathcal{P}_{xy}} f(p)\big(\varphi(x)-\varphi(y)\big)^2$$

$\varphi(x)-\varphi(y) = \sum_{(u,v)\in p}\big(\varphi(v)-\varphi(u)\big)$

$$= \sum_{xy}\sum_{p\in\mathcal{P}_{xy}} f(p)\left(\sum_{(u,v)\in p}\big(\varphi(v)-\varphi(u)\big)\right)^2$$

$$\le \sum_{xy}\sum_{p\in\mathcal{P}_{xy}} f(p)\left(\sum_{(u,v)\in p} 1^2\right)\left(\sum_{(u,v)\in p}\big(\varphi(v)-\varphi(u)\big)^2\right)$$

$$\le \sum_{xy}\sum_{p\in\mathcal{P}_{xy}} f(p)\,|p|\sum_{(u,v)\in p}\big(\varphi(v)-\varphi(u)\big)^2$$

Cauchy Shwartz

$(\sum_i a_i b_i)^2 \le \big(\sum_i a_i^2\big)\big(\sum_i b_i^2\big)$

$$= \sum_{e=(u,v)}\big(\varphi(v)-\varphi(u)\big)^2\sum_{p\ni e} f(p)|p| \quad \text{(和の入れ替え)}$$

$$\le D(f)\sum_{e=(u,v)}\big(\varphi(v)-\varphi(u)\big)^2\sum_{p\ni e} f(p)$$

$$D(f)\sum_{e=(u,v)}\big(\varphi(v)-\varphi(u)\big)^2\sum_{p\ni e}f(p)$$
$$=D(f)\sum_{e=(u,v)}\big(\varphi(v)-\varphi(u)\big)^2 f(e)$$
$$\leq D(f)\rho(f)\sum_{e=(u,v)}\big(\varphi(v)-\varphi(u)\big)^2 Q(e)$$
$$\leq D(f)\rho(f)\sum_{e=(u,v)}\big(\varphi(v)-\varphi(u)\big)^2 \pi(u)P(u,v)$$
$$=2D(f)\rho(f)\mathcal{E}_P(\varphi,\varphi)$$

- $f(e)=\sum_{p\ni e}f(p)$
- $\rho(f)=\max_e\frac{f(e)}{Q(e)}$
- $Q(e)\coloneqq\pi(z)P(z,z')$

例1: Hypercube (uniform flow)

$\{0,1\}^n$上のlazy RW; $N = 2^n$とする．$\pi(x) = \frac{1}{N}$

- $Q(u,v) = \pi(u)P(u,v) = \frac{1}{N}\frac{1}{2n} = \frac{1}{2nN}$
- $R(x,y) = \pi(x)\pi(y) = \frac{1}{N^2}$

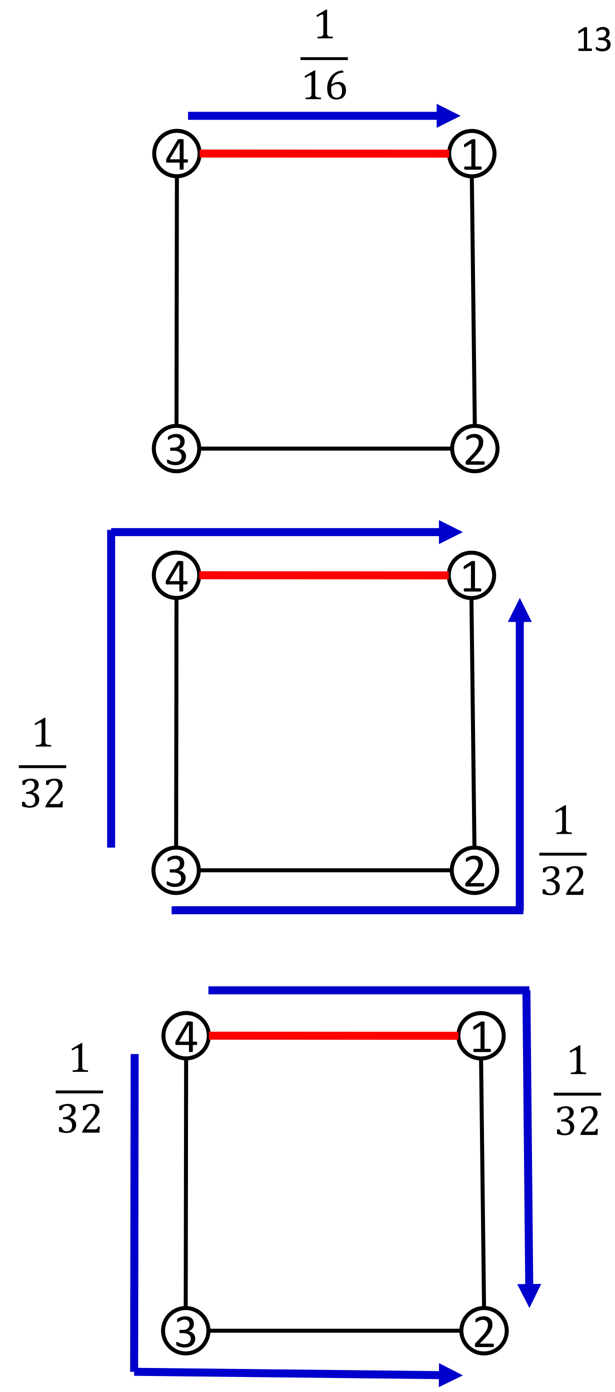
Flowの設計: $R(x,y)$を最短距離に均等に流す

- $f(e) = \frac{\sum_{e \in E} f(e)}{|E|} = \frac{\frac{1}{N^2}\sum_{xy} \mathrm{dist}(x,y)}{Nn} = \frac{\frac{n}{2}}{Nn} = \frac{1}{2N}$
- $\rho(f) = \max_e \frac{f(e)}{Q(e)} = \frac{\frac{1}{2N}}{\frac{1}{2nN}} = n$
- $D(f) = \max_f |p| = n$

- $\frac{1}{1-\lambda_2} \leq n^2$ したがって$\tau \leq \frac{\log\frac{1}{\pi_{\min}}}{1-\lambda_2} \leq n^3 \log 2$

$e = (4,1)$:

$$f_{41}(e) + f_{31}(e) + f_{42}(e) = \frac{1}{16} + \frac{1}{32} + \frac{1}{32} = \frac{4}{32} = \frac{1}{8}$$



例1: Hypercube (uniform flow)

$\{0,1\}^n$上のlazy RW; $N=2^n$とする．$\pi(x)=\frac{1}{N}$

- $Q(u,v)=\pi(u)P(u,v)=\frac{1}{N}\frac{1}{2n}=\frac{1}{2nN}$
- $R(x,y)=\pi(x)\pi(y)=\frac{1}{N^2}$

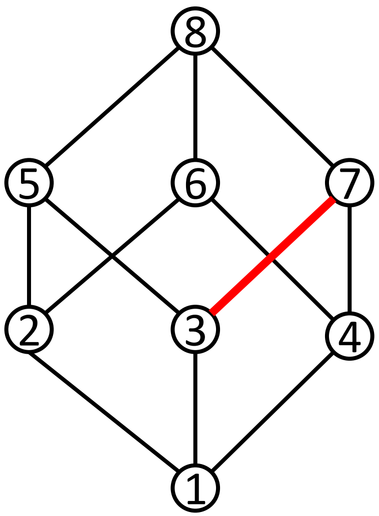
Flowの設計: $R(x,y)$を最短距離に均等に流す

➢ $f(e)=\frac{\sum_{e\in E}f(e)}{|E|}=\frac{\frac{1}{N^2}\sum_{xy}\mathrm{dist}(x,y)}{Nn}=\frac{\frac{n}{2}}{Nn}=\frac{1}{2N}$

➢ $\rho(f)=\max_e\frac{f(e)}{Q(e)}=\frac{\frac{1}{2N}}{\frac{1}{2nN}}=n$

➢ $D(f)=\max_f|p|=n$

- $\frac{1}{1-\lambda_2}\le n^2$ したがって$\tau\le\frac{\log\frac{1}{\pi_{\min}}}{1-\lambda_2}\le n^3\log 2$



$e=(3,7)$:

$f_{17}(e)+f_{18}(e)+f_{27}(e)+f_{34}(e)+f_{36}(e)+f_{37}(e)+f_{38}(e)+f_{54}(e)+f_{57}(e)$

$=\frac{1}{64}\left(\frac{1}{2}+\frac{1}{6}+\frac{2}{6}+\frac{1}{2}+\frac{2}{6}+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{2}\right)=\frac{1}{64}\left(\frac{6}{6}+\frac{4}{2}+1\right)=\frac{1}{16}$

FLOW ENCODING

paths(e) ⊆ paths: eをとおる経路

条件

フローfのencodingはη_e: paths$(e) \to \Omega$で

1. η_eは単射

2. $\exists \beta \leq \mathrm{poly}(n), \forall x, y \in \mathrm{paths}(e)$,

$\pi(x)\pi(y) \leq \beta\pi(x)\pi\big(\eta_e(x,y)\big)$　　ただし$e = (z, z')$

定理 10.4.

条件を満たすエンコードに対して

$$\rho(f) \leq \beta \max_{P(z,z')} \frac{1}{P(z,z')}$$

証明

- $f(e) = \sum_{(x,y)\in\mathrm{paths}(e)} \pi(x)\pi(y)$

  $\leq \beta \sum_{(x,y)\in\mathrm{paths}(e)} \pi(z)\pi\big(\eta_e(x,y)\big) \leq \beta\pi(z)$
- $Q(e) = \pi(z)P(z, z')$
- $\rho(e) = \frac{f(e)}{Q(e)} \leq \beta \frac{1}{P(z,z')}$

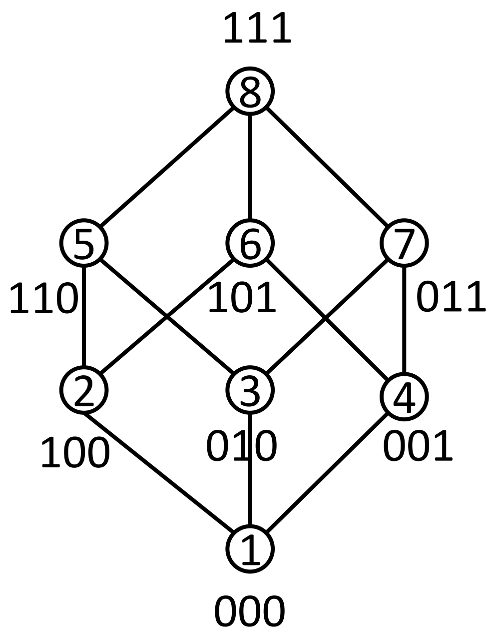
例2: hypercube (canonical path)

Path encoding: Left-right bit-fixing

$\eta_e(x,y) = x_1 x_2, \ldots, x_i, y_{i+1} y_{i+2}, \ldots, y_n$

- $\pi(x)\pi(y) = \pi(z)\pi\big(\eta_e(x,y)\big)$
- $\rho(f) \le \max_{z,z'} \frac{1}{P(z,z')} = 2n$
- $D(f) = n$

よって$\tau \le D(f)\rho(f) \log \frac{1}{\pi_{\min}} = 2n^3 \log 2$



おわり

Warm start

reversible不要

定理

Pはエルゴード的でlazyとする.

$X_0 = x$とするマルコフ連鎖に対して

$$\tau_x(\epsilon) \le \frac{1}{\alpha}\left(2\ln\epsilon^{-1} + \ln\left(4\pi(x)\right)^{-1}\right)$$

ただし$\alpha = \inf_{\varphi \not\equiv c} \frac{\mathcal{E}_P(\varphi,\varphi)}{\mathrm{Var}_\pi[\varphi]}$

$\tau_x(\epsilon) = \min\{t \mid \forall s \ge t, d_{\mathrm{TV}}(P_x^s, \pi) \le \epsilon\}$

ただしP_x^sは初期状態xからs回推移後の分布

固有値を経由せず，直接証明する.

Mark Jerrum, Counting, Sampling and Integrating: Algorithms and Complexity, Lectures in Mathematics. ETH Zuerich, Springer, 2013.